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A TREATISE ON

PLANE TRIGONOMETRY

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NEW YORK : THE MACMILLAN CO.
BOMBAY
CALCUTTA } MACMILLAN AND CO., LTD.
MADRAS
TORONTO : THE MACMILLAN CO.
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TOKYO : MARUZEN-KABUSHIKI-KAISHA

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A TREATISE ON
PLANE TRIGONOMETRY

BY

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SEVENTH EDITION

Cambridge:
at the University Press
1928

First Edition 1891
Second Edition 1897
Third Edition (revised and enlarged) 1911
Fourth Edition 1918
Fifth Edition 1921
Sixth Edition 1925
Seventh Edition 1928

Printed in Great Britain
by Turnbull & Spears, Edinburgh

PREFACE TO THE THIRD EDITION

I HAVE taken the opportunity afforded by the need for a new edition to subject the whole work to a careful revision, and to introduce a considerable amount of new matter. In Chapter I I have inserted a theory of the lengths of circular arcs, and of the areas of circular sectors, based upon arithmetic definitions of their measures. Much of that part of the work which deals with Analytical Trigonometry has been re-written. Proofs of the transcendency of the numbers e and π have been introduced into Chapter XV. It is hoped that the proof there given of the impossibility of "squaring the circle" will prove of interest to many readers to whom a detailed discussion of this very interesting result of modern Analysis has hitherto not been readily accessible.

E. W. HOBSON.

CHRIST'S COLLEGE, CAMBRIDGE,
October, 1911.

PREFACE TO THE FOURTH EDITION

IN this edition a few errors in the text have been corrected.

E. W. HOBSON.

CHRIST'S COLLEGE, CAMBRIDGE,
December, 1917.

PREFACE TO THE FIRST EDITION

IN the present treatise, I have given an account, from the modern point of view, of the theory of the circular functions, and also of such applications of these functions as have been usually included in works on Plane Trigonometry. It is hoped that the work will assist in informing and training students of Mathematics who are intending to proceed considerably further in the study of Analysis, and that, in view of the fulness with which the more elementary parts of the subject have been treated, the book will also be found useful by those whose range of reading is to be more limited.

The definitions given in Chapter III, of the circular functions, were employed by De Morgan in his suggestive work on *Double Algebra and Trigonometry*, and appear to me to be those from which the fundamental properties of the functions may be most easily deduced in such a way that the proofs may be quite general, in that they apply to angles of all magnitudes. It will be seen that this method of treatment exhibits the formulae for the sine and cosine of the sum of two angles, in the simplest light, merely as the expression of the fact that the projection of the hypothenuse of a right-angled triangle on any straight line in its plane is equal to the sum of the projections of the sides on the same line.

The theorems given in Chapter VII have usually been deferred until a later stage, but as they are merely algebraical consequences of the addition theorems, there seemed to be no reason why they should be postponed

A strict proof of the expansions of the sine and cosine of an angle in powers of the circular measure has been given in Chapter VIII; this is a case in which, in many of the text books in use, the passage from a finite series to an infinite one is made without any adequate investigation of the value of the remainder after a finite number of terms, simplicity being thus attained at the expense of rigour. It may perhaps be thought that, at this stage, I might have proceeded to obtain the infinite product formulae for the sine and cosine, and thus have rounded off the theory of the functions of a real angle; for convenience of arrangement, however, and in order that the geometrical applications might not be too long deferred, the investigation of these formulae has been postponed until Chapter XVII.

As an account of the theory of logarithms of numbers is given in all works on Algebra, it seemed unnecessary to repeat it here; I have consequently assumed that the student possesses a knowledge of the nature and properties of logarithms, sufficient for practical application to the solution of triangles by means of logarithmic tables.

In Chapter XII, I have deliberately omitted to give any account of the so-called Modern Geometry of the triangle, as it would have been impossible to find space for anything like a complete account of the numerous properties which have been recently discovered; moreover many of the theorems would be more appropriate to a treatise on Geometry than to one on Trigonometry.

The second part of the book, which may be supposed to commence at Chapter XIII, contains an exposition of the first principles of the theory of complex quantities; hitherto, the very elements of this theory have not been easily accessible to the English student, except recently in Prof. Chrystal's excellent treatise on Algebra. The subject of Analytical Trigonometry has been too frequently presented to the student in the state in which it was left by Euler, before the researches of Cauchy, Abel, Gauss, and others, had placed the use of imaginary quantities and especially the theory of infinite series and products, where real or complex quantities are involved, on a firm scientific basis. In the Chapter on the exponential theorem and logarithms, I have ventured to introduce the term "generalized logarithm" for the doubly infinite series of values of the logarithm of a quantity.

I owe a deep debt of gratitude to Mr W. B. Allcock, Fellow of Emmanuel College, and to Mr J. Greaves, Fellow of Christ's College, for their great kindness in reading all the proofs; their many suggestions and corrections have been an invaluable aid to me. I have also to express my thanks to Mr H. G. Dawson, Fellow of Christ's College, who has undertaken the laborious task of verifying the examples. My acknowledgments are due to Messrs A. and C. Black, who have most kindly placed at my disposal the article "Trigonometry" which I wrote for the *Encyclopædia Britannica.*

During the preparation of the work, I have consulted a large number of memoirs and treatises, especially German and French ones. In cases where an investigation which appeared to be private property has been given, I have indicated the source.

I need hardly say that I shall be very grateful for any corrections or suggestions which I may receive from teachers or students who use the work.

E. W. HOBSON.

CHRIST'S COLLEGE, CAMBRIDGE,
March, 1891.

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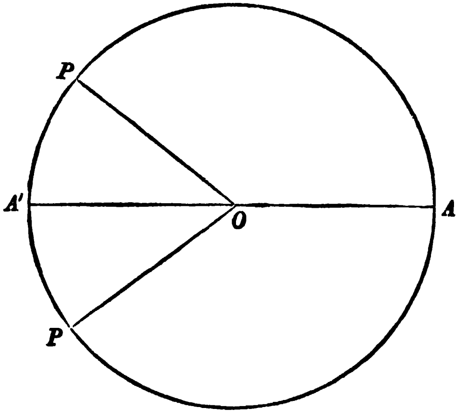
CHAPTER I.

THE MEASUREMENT OF ANGULAR MAGNITUDE.

1. The primary object of the science of Plane Trigonometry is to develope a method of solving plane triangles. A plane triangle has three sides and three angles, and supposing the magnitudes of any three of these six parts to be given, one at least of the three given parts being a side, it is possible, under certain limitations, to determine the magnitudes of the remaining three parts; this is called solving the triangle. We shall find that in order to attain this primary object of the science, it will be necessary to introduce certain functions of an angular magnitude; and Plane Trigonometry, in the extended sense, will be understood to include the investigation of all the properties of these so-called circular functions and their application in analytical and geometrical investigations not connected with the solution of triangles.

The generation of an angle of any magnitude.

2. The angles considered in Euclidean Geometry are all less than two right angles, but for the purposes of Trigonometry it is necessary to extend the conception of angular magnitude so as to include angles of all magnitudes, positive and negative. Let OA be a fixed straight line, and let a straight line OP, initially coincident with OA, turn round the point O in the counter-clockwise direction, then as it turns, it generates the angle AOP; when OP reaches the position OA', it has generated an angle equal to two right angles, and we may suppose it to go on turning in the same direction until it is again coincident with *OA*; it has then turned through four right angles; we may then suppose *OP* to go on



turning in the same direction, and in fact, to make any number of complete turns round *O*; each time it makes a complete revolution it describes four right angles, and if it stop in any position *OP*. it will have generated an angle which may be of any absolute magnitude, according to the position of *P*. We shall make the convention that an angle so described is positive, and that the angle described when *OP* turns in the opposite or clockwise direction is negative. This convention is of course perfectly arbitrary, we might, if we pleased, have taken the clockwise direction for the positive one. In accordance with our convention then, whenever *OP* makes a complete counter-clockwise revolution, it has turned through four right angles reckoned positive, and whenever it makes a complete clockwise revolution, it has turned through four right angles taken negatively.

As an illustration of the generation of angles of any magnitude, we may consider the angle generated by the large hand of a clock. Each hour, this hand turns through four right angles, and preserves no record of the number of turns it has made; this, however, is done by the small hand, which only turns through one-twelfth of four right angles in the hour, and thus enables us to measure the angle turned through by the large hand in any time less than twelve hours. In order that the angles generated by the large hand may be positive, and that the initial position may agree with that in our figure, we must suppose the hands to revolve in the opposite direction to that in which they actually revolve in a clock, and to coincide at three o'clock instead of at twelve o'clock.

3. Supposing OP in the figure to be the final position of the turning line, the angle it has described in turning from the position OA to the position OP may be any one of an infinite number of positive and negative angles, according to the number and direction of the complete revolutions the turning line has made, and any two of these angles differ by a positive or negative multiple of four right angles. We shall call all these angles bounded by the two lines OA, OP *coterminal angles*, and denote them by (OA, OP); the arithmetically smallest of the angles (OA, OP) is the Euclidean angle AOP, and all the others are got by adding positive or negative multiples of four right angles to the algebraical value of this.

The numerical measurement of angles.

4. Having now explained what is meant by an angle of any positive or negative magnitude, the next step to be made, as regards the measurement of angles, is to fix upon a system for their numerical measurement. In order to do this, we must decide upon a unit angle, which may be any arbitrarily chosen angle of fixed magnitude; then all other angles will be measured numerically by the ratios they bear to this unit angle. The natural unit to take would be the right angle, but as the angles of ordinary size would then be denoted by fractions less than unity, it is more convenient to take a smaller angle as the unit. The one in ordinary use is the *degree*, which is one ninetieth part of a right angle. In order to avoid having to use fractions of a degree, the degree is subdivided into sixty parts called *minutes*, and the minute into sixty parts called *seconds*. Angles smaller than a second are denoted as decimals of a second, the *third*, which would be the sixtieth part of a second, not being used. An angle of d degrees is denoted by d°, an angle of m minutes by m', and an angle of n seconds by n'', thus an angle $d^\circ\, m'\, n''$ means an angle containing d degrees $+\, m$ minutes $+\, n$ seconds, and is equal to $\frac{d}{90}+\frac{m}{90\,.\,60}+\frac{n}{90\,.\,60\,.\,60}$ of a right angle.

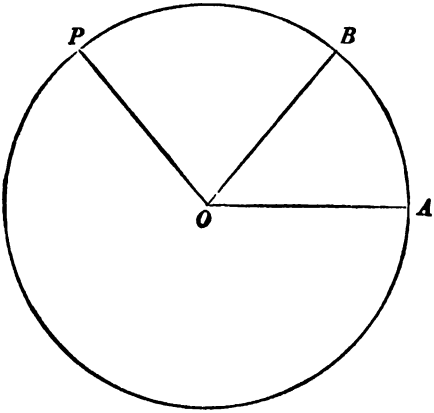
This system of numerical measurement of angles is called the sexagesimal system. For example, the angle $23^\circ\, 14'\, 56''\cdot 4$ denotes $\frac{23}{90}+\frac{14}{90\,.\,60}+\frac{56\cdot 4}{90\,.\,60\,.\,60}$ of a right angle.

It has been proposed to use the decimal system of measurement of angles. In this system the right angle is divided into a hundred grades, the grade into a hundred minutes, and the minute into a hundred seconds; an angle of g grades, m minutes and n seconds is then written $g^{g}\, m^{\backprime}\, n^{\backprime\backprime}$. For example, the angle $13^{g}\, 97^{\backprime}\, 4^{\backprime\backprime}\cdot 2$ is equal to $13\cdot 97042$ of a right angle. This system has however never come into use, principally because it would be inconvenient in turning time into grades of longitude, unless the day were divided differently than it is at present. The day might, if the system of grades were adopted, be divided into forty hours instead of twenty-four, and the hour into one hundred minutes, thus involving an alteration in the chronometers; one of our present hours of time corresponds to a difference of 50/3 grades of longitude, which being fractional is inconvenient.

It is an interesting fact that the division of four right angles into 360 parts was used by the Babylonians; there has been a good deal of speculation as to the reason for their choice of this number of subdivisions.

The circular measurement of angles.

5. Although, for all purely practical purposes, the sexagesimal system of numerical measurement of angles is universally used, for theoretical purposes it is more convenient to take a different unit angle. In any circle of centre O, suppose AB to be an arc



whose length is equal to the radius of the circle; we shall shew that the angle AOB is of constant magnitude independent of the particular circle used; this angle is called the ***Radian*** or unit of circular measure, and the magnitude of any other angle is expressed by the ratio which it bears to this unit angle, this ratio being called the *circular measure of the angle.*

6. In order to shew that the Radian is a fixed angle, we shall assume the following two theorems:

(*a*) In the same circle, the lengths of different arcs are to one another in the same ratio as the angles which those arcs subtend at the centre of the circle.

(*b*) The length of the whole circumference of a circle bears to the diameter a ratio which is the same for all circles.

The theorem (*a*) is contained in Euclid, Book VI. Prop. 33, and we shall give a proof of the theorem (*b*) at the end of the present Chapter. From (*a*) it follows that

$$\frac{\text{arc } AB}{\text{circumference of the circle}} = \frac{\angle AOB}{4 \text{ right angles}}.$$

Since the arc AB is equal to the radius of the circle, the first of these ratios is, according to (*b*), the same in all circles, consequently the angle AOB is of constant magnitude independent of the particular circle used.

7. It will be shewn hereafter that the ratio of the circumference of a circle to its diameter is an irrational number; that is, we are unable to give any integers m and n such that m/n is exactly equal to the ratio. We shall, in a later Chapter, give an account of the various methods which have been employed to calculate approximately the value of this ratio, which is usually denoted by π. At present it is sufficient to say that π can only be obtained in the form of an infinite non-recurring decimal, and that its value to the first twenty places of decimals is

$$3{\cdot}14159265358979323846.$$

For many purposes it will be sufficient to use the approximate value 3·14159. The ratios $\frac{22}{7} = 3{\cdot}\dot{1}4285\dot{7}$, $\frac{355}{113} = 3{\cdot}1415929\ldots$ may be used as approximate values of π, since they agree with the correct value of π to two and six places of decimals respectively.

8. We have shewn that the radian is to four right angles in the ratio of the radius to the circumference of a circle; the radian is therefore $\frac{2}{\pi} \times$ a right angle; remembering then that a right angle is 90°, and using the approximate value of π, 3·1415927, we obtain for the approximate value of the radian in degrees, $57^\circ{\cdot}2957796$, or reducing the decimal of a degree to minutes and seconds, $57^\circ\, 17'\, 44''{\cdot}81$.

The value of the radian has been calculated by Glaisher to 41 places of decimals of a second[1]. The value of $1/\pi$ has been obtained to 140 places of decimals[2].

9. The circular measure of a right angle is $\frac{1}{2}\pi$, and that of two right angles is π; and we can now find the circular measure of an angle given in degrees, or vice versa; if d be the number of degrees in an angle of which the circular measure is θ, we have $\frac{\theta}{\pi} = \frac{d}{180}$, for each of these ratios expresses the ratio of the given angle to two right angles; thus $\frac{\pi}{180} d$ is the circular measure of an angle of d degrees, and $\frac{180}{\pi}\theta$ is the number of degrees in an angle whose circular measure is θ; if an angle is given in degrees, minutes, and seconds, as $d^\circ\, m'\, n''$, its circular measure is

$$(d + m/60 + n/3600)\,\pi/180.$$

The circular measure of 1° is $\cdot01745329\ldots$, of $1'$ is $\cdot0002908882\ldots$, and that of $1''$ is $\cdot000004848137\ldots\ldots$

10. The circular measure of the angle AOP, subtended at the centre of a circle by the arc AP, is equal to $\frac{\text{arc } AP}{\text{radius of circle}}$, for this ratio is equal to $\frac{\text{arc } AP}{\text{arc } AB}$ or $\frac{\angle AOP}{\angle AOB}$.

The arc AP may be greater than the whole circumference and may be measured either positively, or negatively, according to the direction in which it is measured from the starting point A, so that the circular measure of an angle of any magnitude is the measure of the arc which subtends the angle, divided by the radius of the circle. The length of an arc of a circle of radius r is $r\theta$, where θ is the circular measure of the angle the arc subtends at the centre of the circle. The whole circumference of the circle is therefore $2\pi r$.

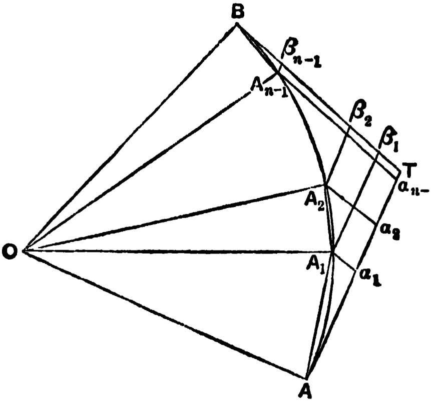
---

[1] On the calculation of the value of the theoretical unit angle to a great number of places. *Quarterly Journal*, Vol. IV.

[2] See Grunert's *Archiv*, Vol. I., 1841.

The length of a circular arc.

11. It has been assumed above that the length of a circular arc is a definite conception, and that it is capable of numerical measurement; this matter will be now investigated. The primary notion of length is that of a linear interval, or finite portion of a straight line; and the notion of the length of an arc of a curve, for example of a circular arc, must be regarded as derivative. That a given finite portion of a straight line has a length which can be represented by a definite rational or irrational number, dependent upon an assumed unit of length, will be here taken for granted. In order to define the length of a circular arc AB, we proceed as follows: Let a number of points of division $A_1, A_2, \dots A_{n-1}$ of the arc AB be assigned, and consider the unclosed polygon



$$AA_1A_2 \dots A_{n-1}B;$$

the sum of the lengths of the sides $AA_1 + A_1A_2 + \dots + A_{n-1}B$ of this polygon has a definite numerical value p_1. Next let a new polygon $AA_1'A_2' \dots A_{n'-1}'B$, where $n' > n$, be inscribed in the arc AB, the greatest side of this polygon being less than the greatest side of $AA_1A_2 \dots B$; let the sum of the sides of this new unclosed polygon be p_2. Proceeding further by successive subdivision of the arc AB, we obtain a sequence of inscribed unclosed polygons of which the lengths are denoted by the numbers $p_1, p_2, \dots p_n, \dots$ of a sequence which may be continued indefinitely. In case the number p_n has a definite limit l, independent of the mode of the successive sub-divisions of the arc AB, that mode being subject only to the condition that the greatest side of the unclosed polygon corresponding to p_n becomes indefinitely small as n is indefinitely increased, then the arc AB is said to have the length l. In order to shew that a circular arc has a length, it is necessary to shew that this limit l exists, and this we proceed to do. It is clear from the definition that, if ABC be an arc, then if AB, BC have definite lengths, so also has AC; and that the length of ABC is the sum of the lengths of the arcs AB, BC. It will therefore be sufficient to shew that an arc which is less than a semicircle has a definite length. In the first place we consider a particular sequence of polygons such that the corners of each polygon are also corners of all the subsequent polygons of the sequence. Denoting by $P_1, P_2, \ldots P_n, \ldots$ the lengths of these unclosed polygons, it can be shewn that

$$P_1 < P_2 < P_3, \ldots < P_n, \ldots;$$

for, by elementary geometry, it is seen that A_rA_{r+1} is less than the sum of the sides of an unclosed polygon which joins A_r, A_{r+1}. Again all the numbers $P_1, P_2, \ldots P_n, \ldots$ are less than a fixed number. For let TA, TB be the tangents at A, B the ends of the arc, and draw $A_1\alpha_1, A_2\alpha_2, \ldots A_{n-1}\alpha_{n-1}$ parallel to BT, and also draw $A_1\beta_1, A_2\beta_2, \ldots A_{n-1}\beta_{n-1}$ parallel to AT. We have then

$$AA_1 < A\alpha_1 + A_1\alpha_1 < A\alpha_1 + T\beta_1, \text{ and } A_1A_2 < \alpha_1\alpha_2 + \beta_1\beta_2, \text{ \&c.};$$

hence $$AA_1 + A_1A_2 + \ldots + A_{n-1}B < AT + BT,$$

therefore $$P_n < AT + BT.$$

In accordance with a fundamental principle in the theory of limits, since the sequence $P_1, P_2, \ldots P_n, \ldots$ of numbers is such that each one is less than the next one, and such that all of them are less than a fixed number, the sequence has a limit l, which is such that, if ϵ be an arbitrarily chosen positive number, as small as we please, from and after some value n_1 of n, all the numbers P_n differ from l by less than ϵ.

To shew that if $p_1, p_2, \ldots p_n, \ldots$ are the lengths of any sequence of unclosed polygons whatever joining A, B, not subject to the condition that the corners of each polygon are also corners of all the subsequent ones, but subject only to the condition that the greatest side of the nth polygon decreases as n increases, and has zero for its limit, we compare such a sequence with the special sequence considered above, and which has been shewn to have a definite number l as the limit of the lengths of the polygons Consider a polygon $AA_1A_2 \ldots A_{r-1}B$ of the sequence whose lengths are $P_1, P_2, \ldots$, so far advanced that its length is greater than $l-\epsilon$. An integer n' can be determined, such that, if $n \geqq n'$, the polygon $A\alpha\beta\gamma, \ldots \kappa, \ldots B$ of which the length is p_n has its greatest side less than the least side of $AA_1A_2 \ldots A_{r-1}B$ and also less than $\epsilon/2r$. Some of the points α, β, $\gamma, \ldots$ are then in each of the arcs AA_1, $A_1A_2, \ldots$. Let α, β, γ be in AA_1; then

$$A\alpha + \alpha\beta + \beta\gamma + \gamma A_1 > AA_1.$$

Using this and the similar inequalities $A_1\delta + \delta\epsilon + \ldots + \kappa A > A_1A_2$, we have by addition, and remembering that γA_1, $A_1\delta, \ldots$ are all less than $\epsilon/2r$, $p_n + \epsilon > AA_1 + A_1A_2 + \ldots + A_{r-1}B > l - \epsilon$, therefore $p_n > l - 2\epsilon$, provided $n \geqq n'$. Next consider a polygon $AA_1'A_2'A_3' \ldots B$, of the sequence whose lengths are $P_1, P_2, \ldots$, so far advanced that the greatest side is less than the least side of $A\alpha\beta\gamma \ldots \kappa B$, and also less than $\epsilon/2s$, where s is the number of sides in this latter polygon; as before we see that

$$p_n < \epsilon + AA_1' + A_1'A_2' + \ldots < l + \epsilon.$$

It has now been shewn that, if $n \geqq n'$, p_n lies between $l+\epsilon$ and $l-2\epsilon$, and therefore differs from l by less than 2ϵ. Since ϵ is arbitrarily chosen, and to each value of it there corresponds an integer n', it has been shewn that p_n has the same limit l, when n is indefinitely increased, as for the special sequence of polygons first considered.

It has now been shewn that the length of a circular arc is measured by a definite number, a unit of length being assumed.

The circumference C of the whole circle is itself given as the limit of the perimeters of a sequence of inscribed closed polygons, such that the greatest of the sides becomes indefinitely small as the sequence proceeds.

That the lengths of different arcs of the same circle are to one another as the angles subtended by those arcs at the centre of the circle may now be established as in Euclid, Book VI. Prop. 33.

To prove that the circumferences of circles vary as their diameters, let us consider two circles of which the diameters are d and d'. If two similar polygons be inscribed in the circles, it follows from the properties of similar rectilineal figures that the perimeters of these polygons are to one another in the ratio of d

to d'. The circumferences C and C' of the circles may be taken to be the limits of the perimeters p_n, p_n' of two sequences of polygons such that the polygon corresponding to p_n is for each value of n similar to the polygon corresponding to p_n'. Since $p_n : p_n' = d : d'$, it follows that the ratio of the limit of p_n to that of the limit of p_n' is equal to the ratio of $d : d'$; and therefore

$$C : C' = d : d'$$

The area of a sector of a circle.

12. The area of the sector OAB of a circle, with centre O, bounded by the arc AB is defined to be the limit of the sum of the areas of the triangles OAA_1, $OA_1A_2, \ldots OA_{n-1}B$, when the number of sides of the polygon $AA_1A_2, \ldots B$ is increased indefinitely and the greatest of its sides is diminished indefinitely, as explained in § 11. It must be proved that this limit exists as a definite number.

Let $q_1, q_2, \ldots q_n$ be the lengths of the perpendiculars from O on the sides AA_1, $A_1A_2, \ldots A_{n-1}B$; then the sum of the areas of the triangles is $\frac{1}{2}(q_1 . AA_1 + q_2 . A_1A_2 + \ldots + q_n . A_{n-1}B)$, and this lies between $\frac{1}{2}q' . p_n$ and $\frac{1}{2}q''p_n$; where q' and q'' are the greatest and least of the numbers $q_1, q_2, \ldots q_n$, and p_n is the sum of the sides of the polygon. The limit of p_n exists as the length of the arc AB; also the two numbers q', q'' have one and the same limit, the radius of the circle, since they differ from this radius by less than half the greatest side of the polygon. Therefore the area of the sector is a definite number, equal to half the product of the radius r of the circle, and the length $r\theta$ of the arc AB; where θ is the circular measure of the angle AOB. Thus area $AOB = \frac{1}{2}r^2\theta$. The whole circle is a sector of which the bounding arc is the whole circumference; hence the area of the whole circle is πr^2.

EXAMPLES ON CHAPTER I.

1. What must be the unit of measurement, that the numerical measure of an angle may be equal to the difference between its numerical measures as expressed in degrees and in circular measure?

2. If the measures of the angles of a triangle referred to 1°, 100′, 10000″ as units be in the proportion of 2, 1, 3, find the angles.

3. Find the number of degrees in an angle of a regular polygon of n sides (1) when it is convex, (2) when its periphery surrounds the inscribed circle m times.

4. Two of the angles of a triangle are $52^\circ\, 53'\, 51''$, $41^{g}\, 22^{\backprime}\, 50^{\backprime\backprime}$ respectively; find the third angle.

5. Find, to five decimal places, the arc which subtends an angle of 1° at the centre of a circle whose radius is 4000 miles.

6. An angle is such that the difference of the reciprocals of the number of grades and degrees in it is equal to its circular measure divided by 2π; find the angle.

7. The angles of a plane quadrilateral are in A.P. and the difference of the greatest and least is a right angle; find the number of degrees in each angle and also the circular measure.

8. In each of two triangles the angles are in G.P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is 240°; find the circular measure of the angles.

9. If an arc of 10 feet on a circle of eight feet diameter subtend at the centre an angle $143^\circ\, 14'\, 22''$, find the value of π to four decimal places.

10. Find two regular figures such that the number of degrees in an angle of the one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second.

11. ABC is a triangle such that, if each of its angles in succession be taken as the unit of measurement, and the measures formed of the sums of the other two, these measures are in A.P. Shew that the angles of the triangle are in H.P. Also shew that only one of these angles can be greater than $\frac{2}{3}$ of a right angle.

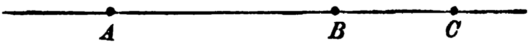
12. Shew that there are eleven and only eleven pairs of regular polygons which are such that the number of degrees in an angle of one of them is equal to the number of grades in an angle of the other, and that there are only four pairs in which these angles are expressed by integers.

13. The apparent angular diameter of the sun is half a degree. A planet is seen to cross its disc in a straight line at a distance from its centre equal to three-fifths of its radius. Prove that the angle subtended at the earth, by the part of the planet's path projected on the sun, is $\pi/450$.

CHAPTER II.

THE MEASUREMENT OF LINES. PROJECTIONS.

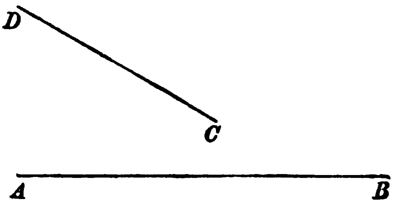
13. If it is required to measure a given length along a given straight line, supposed indefinitely prolonged in both directions, starting from any assumed point, the question arises, in which direction is the given length to be measured off. In order to avoid ambiguity, we agree that to lengths measured along the straight line in one direction a positive number shall be assigned, and consequently in the other direction a negative number; it is necessary then in such a straight line to assign the positive direction. Suppose, in the figure, we agree that lines measured



from left to right shall be considered to have a positive measure; the length AB is then measured positively, and the length BA negatively, or $AB = -BA$.

14. If C be any third point anywhere on the straight line, we shall have $AB = AC + CB$, for example if, as in the figure, C lies beyond B, the line CB is negative, and therefore its numerical length is subtracted from that of AC. The sum of the measures of the lengths of any number of such straight lines generated by a point which starts at A and finishes its motion at B is accordingly equal to that of AB.

15. When, as in Art. 2, an angle is generated by a straight line OP turning from an initial position OA, we shall suppose that, whilst turning, the positive direction in the line OP remains unaltered, thus the angle which has been generated in any position of OP is the angle between the two positive directions of the bounding lines. It follows, that if AB, CD are the positive directions in two straight lines, the angle between AB and DC differs by two right angles from the angle between AB and CD, for a line revolving from the position AB must turn through an angle, in order to coincide with DC, 180° greater or less than the angle it must turn through in order to coincide with CD.



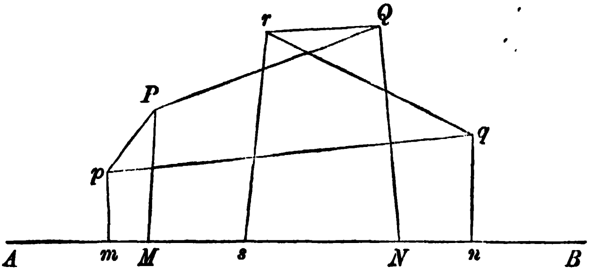
If we consider all the coterminal angles bounded by AB and CD, and by AB and DC, respectively, we shall have $(AB, CD) = (AB, DC) + 180°$, the angles being all measured in degrees.

16. When a straight line moves parallel to itself, we shall suppose its positive direction to be unaltered, so that if AB, CD are non-intersecting straight lines, the angle between them is equal to the angle between AB and a straight line drawn through A parallel to CD. For ordinary geometrical purposes, the angle between AB and CD is the smallest angle between AB and this parallel, irrespective of sign.

Projections.

17. If from the extremities P, Q of any straight line PQ perpendiculars PM, QN be drawn to any straight line AB, the portion MN, with its proper sign, is called the projection of the straight line PQ on the straight line AB. It should be noticed that PQ and AB need not necessarily be in the same plane. The projection of QP is NM, and has therefore the opposite sign to that of PQ.

If the points P and Q be joined by any broken line, such as $PpqrQ$, the sum of the projections of Pp, pq, qr, rQ on AB is equal



to the projection of PQ on AB. For the sum of the projections is $Mm + mn + ns + sN$, which is, according to Art. 14, equal to MN. We obtain thus the fundamental property of projections. *The sum of the projections on any fixed straight line, of the parts of any broken line joining two points P and Q, depends only upon the positions of P and Q, being independent of the manner in which P and Q are joined.*

A particular case of this proposition is the following:

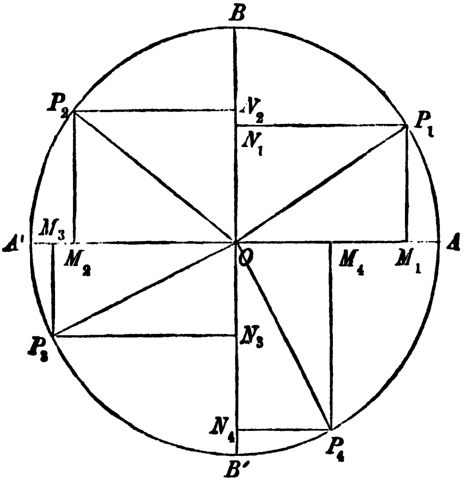
The sum of the projections on any straight line, of the sides, taken in order, of any closed polygon, is zero. If, in the above figure, the points P and Q coincide, the broken line joining them becomes a closed polygon, and since the projection of PQ is zero, the sum of the projections of the sides, taken in order, of the polygon, is also zero. The polygon is not necessarily plane, and may have any number of re-entrant angles.

CHAPTER III.

THE CIRCULAR FUNCTIONS.

Definitions of the circular functions.

18. HAVING now explained the manner in which angular and linear magnitudes are measured, we are in a position to define the Circular Functions or Trigonometrical Ratios. Suppose an angle AOP of any magnitude A, to be generated as in Art. 2, by the revolution of OP from the initial position OA, remembering the convention made as to the sign of angles. Let $B'OB$ be drawn perpendicular to $A'OA$; we suppose the positive directions in $A'OA$ and $B'OB$ to be from O to A and O to B respectively. We



also remember the convention made in Art. 15, as to the positive direction of the revolving line.

The ratio of the projection of OP *on the initial line, to the length* OP, *is called the cosine of the angle* A, *and is denoted by cos* A.

The ratio of the projection of OP *on the straight line* OB *which makes an angle* + 90° *with the initial line, to the length* OP. *is called the sine of the angle* A, *and is denoted by sin* A.

The ratio of the projection of OP *on* OB, *to its projection on* OA, *is called the tangent of the angle* A, *and is denoted by tan* A.

The ratio of the projection of OP *on* OA, *to its projection on* OB, *is called the cotangent of the angle* A, *and is denoted by cot* A.

The ratio of OP, *to its projection on* OA, *is called the secant of the angle* A, *and is denoted by sec* A.

The ratio of OP, *to its projection on* OB, *is called the cosecant of the angle* A, *and is denoted by cosec* A.

Thus we have

$$\cos A = \frac{OM}{OP}, \quad \sin A = \frac{ON}{OP}, \quad \tan A = \frac{ON}{OM},$$

$$\cot A = \frac{OM}{ON}, \quad \sec A = \frac{OP}{OM}, \quad \operatorname{cosec} A = \frac{OP}{ON}.$$

When each of the lengths in the ratios is taken with its proper sign, the sign of OP is always positive, but those of OM, ON are each positive or negative according to the magnitude of the angle A. It should be observed that MP is equal to, and of the same sign as ON, so that

$$\sin A = \frac{MP}{OP}, \quad \tan A = \frac{MP}{OM}, \quad \cot A = \frac{OM}{MP}, \quad \operatorname{cosec} A = \frac{OP}{MP}.$$

In the figure, the angle A has four different magnitudes AOP_1, AOP_2, AOP_3, AOP_4, corresponding to the four positions P_1, P_2, P_3, P_4 of P.

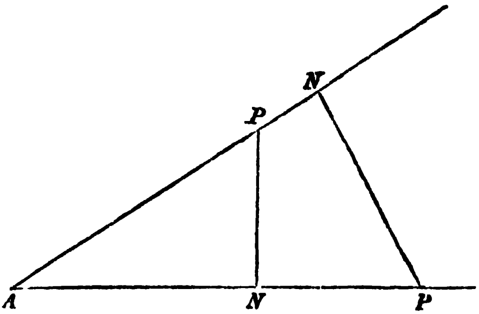
The projection of any positive or negative length AB, on a straight line CD, is obtained by multiplying the length AB taken with its proper sign by the cosine of the angle between the positive directions of the lines on which AB and CD lie; the projection is thus given with its proper sign.

It should be observed that since OP, in the figure, always retains the positive sign as it revolves from the position OA, when it coincides with OA' it has the opposite sign to that of OA'.

19. The six ratios defined above are the six Circular Functions, called also Trigonometrical Ratios or Trigonometrical Functions. Each of them depends only upon the magnitude of the angle A, and not upon the absolute length of OP. This follows from the property of similar triangles, that the ratios of the sides are the same in all similar triangles, so that when OP is taken of a different length, we have the same ratios as before for the same angle. These six ratios are then functions of the angular magnitude A only; we may suppose A to be measured either in the sexagesimal system or in circular measure. For convenience, we shall in general use large letters $A, B, C, \ldots$ for angles measured in degrees, minutes and seconds, and small letters $\alpha, \beta, \theta, \phi, \ldots$ for angles measured in circular measure; so that, for example, $\sin A$ denotes the sine of the angle of which A is the measure in degrees, minutes and seconds, and $\sin \alpha$ is the sine of the angle of which α is the circular measure. To these six circular functions two others may be added, which are sometimes used, the *versine* written $\text{versin}\, A$, and the *coversine* written $\text{coversin}\, A$; these are defined by the equations $\text{versin}\, A = 1 - \cos A$, $\text{coversin}\, A = 1 - \sin A$.

The versine and coversine are used very little in theoretical investigations, but the versine occurs very frequently in the formulae used in navigation.

20. In the case of an acute angle, the definitions of the circular functions may be put into the following form. Let P



be any point in either of the bounding lines of the given angle; draw PN perpendicular to the other bounding line, we have then the right-angled triangle *PAN*, of which the angle *PAN* is the given one *A*.

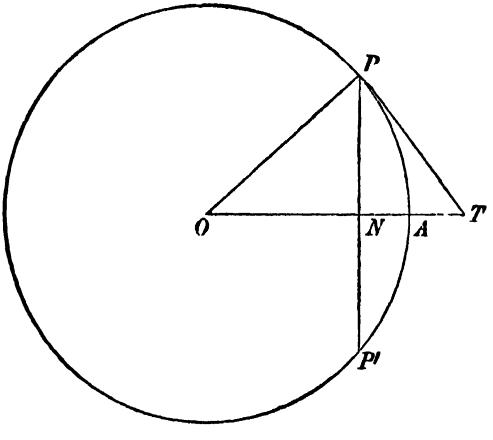
Cos A is then defined as

$$\frac{\textit{side adjacent to } A}{\textit{hypothenuse}}, \quad \sin A \text{ as } \frac{\textit{side opposite to } A}{\textit{hypothenuse}},$$

$$\tan A \text{ as } \frac{\textit{side opposite to } A}{\textit{side adjacent to } A}, \quad \cot A \text{ as } \frac{\textit{side adjacent to } A}{\textit{side opposite to } A},$$

$$\sec A \text{ as } \frac{\textit{hypothenuse}}{\textit{side adjacent to } A}, \quad \operatorname{cosec} A \text{ as } \frac{\textit{hypothenuse}}{\textit{side opposite to } A}.$$

21. Until recently, the circular functions of an angle were defined, not as ratios, but as lengths having reference to arcs of a circle of specified size. If *PA* be an arc of a given circle, let *PN* be drawn perpendicular to *OA*, and let



PT be the tangent at *P*; the line *PN* was defined to be the sine of the arc *PA*, *ON* to be its cosine, *PT* its tangent, *OT* its secant, and *AN* its versine. In this system the magnitudes of the sine, cosine, tangent, &c. depended not only upon the angle *POA*, but also upon the radius of the circle, which had therefore to be specified. The advantage of the present mode of definition of the functions as ratios, is that they are independent of the radius of any circle, and are therefore functions of an angular magnitude only. The sine of an arc was first used by the Arabian Mathematician Al-Battâni (878—918); the Greek Mathematicians had used the chords *PP′* of the double arc, instead of the sine *PN* of the arc *PA*.

Relations between the circular functions.

22. Referring to the definitions of the circular functions, we see at once that there are the following relations between them,

(1) $\cos A \sec A = 1$, (3) $\tan A \cot A = 1$,

(2) $\sin A \operatorname{cosec} A = 1$, (4) $\left.\begin{aligned}\tan A &= \sin A/\cos A\\ \cot A &= \cos A/\sin A\end{aligned}\right\}$.

Expressed in words, the relations (1), (2), (3) assert the facts that the secant, cosecant, and cotangent of an angle are the reciprocals of the cosine, sine, and tangent of the angle respectively; and relation (4) expresses the fact that the tangent of an angle is the ratio of its sine to its cosine, or what, in virtue of (3), comes to the same thing, that the cotangent of an angle is the ratio of the cosine to the sine of the angle.

23. Referring to the figure in Art. 18, the square on OP is by the Pythagoraean theorem, equal to the sum of the squares of its projections OM and MP, so that since the ratios of these projections to OP are the cosine and sine respectively of the angle A, we have $(\cos A)^2 + (\sin A)^2 = 1$ or, as it is usually written, $\cos^2 A + \sin^2 A = 1$. If we divide both sides of this equation by $\cos^2 A$ and remember the relations (1) and (4), we have $1 + \tan^2 A = \sec^2 A$; similarly if we divide both sides of the equation by $\sin^2 A$, and remember the relations (2) and (4), we have $1 + \cot^2 A = \operatorname{cosec}^2 A$. Thus the three identities,

$$\left.\begin{aligned}\cos^2 A + \sin^2 A &= 1\\ 1 + \tan^2 A &= \sec^2 A\\ 1 + \cot^2 A &= \operatorname{cosec}^2 A\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

are different forms of the same relation between the functions.

24. The five independent relations just obtained between the six circular functions enable us to express any five of these functions in terms of the sixth. The student should verify the correctness of the following table, in which the meaning of x in each column stands at the head of that column, and the value of the expressions in each horizontal line, at the beginning.

	$\sin A = x$	$\cos A = x$	$\tan A = x$	$\cot A = x$	$\sec A = x$	$\operatorname{cosec} A = x$
$\sin A =$	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{\sqrt{x^2-1}}{x}$	$\frac{1}{x}$
$\cos A =$	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{1+x^2}}$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{1}{x}$	$\frac{\sqrt{x^2-1}}{x}$
$\tan A =$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{x}$	$\sqrt{x^2-1}$	$\frac{1}{\sqrt{x^2-1}}$
$\cot A =$	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	x	$\frac{1}{\sqrt{x^2-1}}$	$\sqrt{x^2-1}$
$\sec A =$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{1+x^2}$	$\frac{\sqrt{1+x^2}}{x}$	x	$\frac{x}{\sqrt{x^2-1}}$
$\operatorname{cosec} A =$	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{1+x^2}}{x}$	$\sqrt{1+x^2}$	$\frac{x}{\sqrt{x^2-1}}$	x

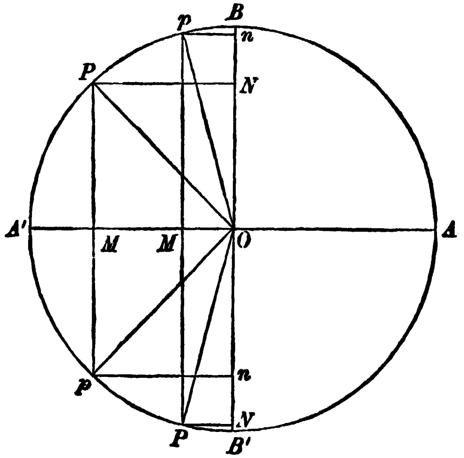
In this table the ambiguities in the signs of the square roots are left undetermined. As an example of the verification of this table we will suppose $\sec A = x$, to be given; we have at once from (1) in Art. 22, $\cos A = 1/x$, and from the second form of (5), $\tan A = \sqrt{x^2-1}$, and then from (3), $\cot A = 1/\sqrt{x^2-1}$; from the first form of (5), $\sin A = \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$; then from (2), $\operatorname{cosec} A = \frac{x}{\sqrt{x^2-1}}$; we have thus verified the correctness of the fifth column in the table.

Range of values of the circular functions.

25. The projection of one straight line upon another cannot be of greater length than the projected line, hence the sine or the cosine of an angle cannot be numerically greater than unity; each of them may have any value between + 1 and − 1, both inclusive; and secant and cosecant which are the reciprocals of the cosine and sine cannot therefore lie between the limits ± 1, and are therefore numerically greater than, or equal to, unity. The tangent or the cotangent, being the ratio of two projections, one of which has its greatest numerical value when the other one vanishes, may have any value between $\pm\infty$. The versine may have any value between

Properties of the circular functions.

26. If the angles AOP, AOp be A and $-A$ respectively, we see that OP and Op have equal projections OM, upon OA, but



that their projections ON, On, on OB, are of equal magnitude but opposite sign, therefore

$$\cos(-A) = \cos A, \text{ and } \sin(-A) = -\sin A \text{ ......(6)};$$

it follows that $\tan(-A) = -\tan A$, $\cot(-A) = -\cot A$,

$$\sec(-A) = \sec A, \quad \operatorname{cosec}(-A) = -\operatorname{cosec} A.$$

If a function of a variable has its magnitude unaltered when the sign of the variable is changed, that function is called an even function, but if the function has the same numerical value as before, but with opposite sign, then that function is called an odd function; for instance x^2 is an even function of x, x^3 is an odd function of x, but $x^2 + x^3$ is neither even nor odd, since its numerical value changes when the sign of x is changed. We see then that *the cosine and the secant of an angle are even functions, and the sine, tangent, cotangent, and cosecant are odd functions. The versine is an even function, but the coversine is neither even nor odd.*

27. The values of the circular functions of an angle depend only upon the position of the bounding line OP, with reference to the other bounding line OA, consequently all the coterminal angles (OA, OP) have the same circular functions, or in other words, all the angles $n \cdot 360° + A$, where n is any positive or negative integer, have their circular functions the same as those of A. If α be the circular measure of the angle which contains A degrees, all the angles $2n\pi + \alpha$, in circular measure, have the same circular functions. We have also, since all the angles $2n\pi - \alpha$ have the same circular functions,

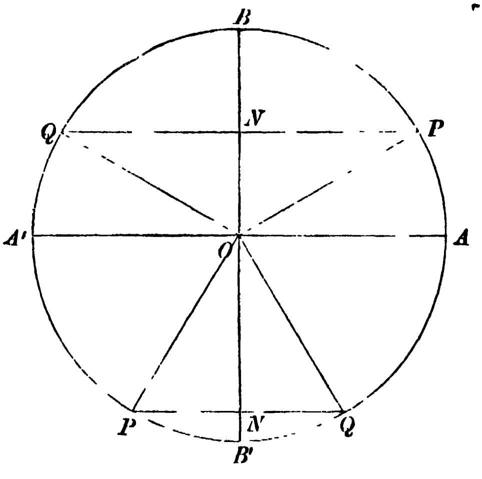
$$\sin(2n\pi - \alpha) = \sin(-\alpha) = -\sin\alpha,$$

and $$\cos(2n\pi - \alpha) = \cos(-\alpha) = \cos\alpha.$$

The properties we have obtained are both included in the equations

$$\left.\begin{aligned}\sin(2n\pi \pm \alpha) &= \pm\sin\alpha\\ \cos(2n\pi \pm \alpha) &= \cos\alpha\end{aligned}\right\} \qquad \dots\dots\dots\dots\dots\dots(6).$$

28. If the angle $180° - A$ or $\pi - \alpha$ is bounded by OQ, then OQ makes the same angle with OA' as OP does with OA, and we see that the projections of OP and OQ on OA are equal and of opposite



sign, and the projections of OP and OQ, on OB, are equal and of the same sign, therefore $\sin(\pi - \alpha) = \sin\alpha$, and $\cos(\pi - \alpha) = -\cos\alpha$. These equations hold whatever α may be, so that we can change α into $-\alpha$, and we have

$$\sin(\pi + \alpha) = \sin(-\alpha) = -\sin\alpha$$

and $$\cos(\pi + \alpha) = -\cos(-\alpha) = -\cos\alpha.$$

Thus we have the system of equations

$$\left.\begin{aligned}\sin(\pi\pm\alpha)&=\mp\sin\alpha\\ \cos(\pi\pm\alpha)&=-\cos\alpha\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

from these we obtain

$$\tan(\pi\pm\alpha)=\pm\tan\alpha\quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Also

$$\left.\begin{aligned}\sin(\overline{2n+1}\,\pi\pm\alpha)&=\sin(\pi\pm\alpha)=\mp\sin\alpha\\ \cos(\overline{2n+1}\,\pi\pm\alpha)&=\cos(\pi\pm\alpha)=-\cos\alpha\\ \tan(\overline{2n+1}\,\pi\pm\alpha)&=\tan(\pi\pm\alpha)=\pm\tan\alpha\end{aligned}\right\}\quad\ldots\ldots(9).$$

29. In the figure of Art. 28, the angle *OP* makes with *OB′* is $90^\circ+A$; therefore the cosine of the angle $90^\circ+A$ or $\frac{1}{2}\pi+\alpha$ is the ratio of the projection of *OP* on *OB′* to *OP*; hence since the projection on *OB′* is equal with opposite sign to the projection on *OB*, we have $\cos(\frac{1}{2}\pi+\alpha)=-\sin\alpha$; changing $\frac{1}{2}\pi+\alpha$ into α, we have $\cos\alpha=-\sin(\alpha-\frac{1}{2}\pi)$, hence in virtue of (6) we have

$$\cos\alpha=\sin(\tfrac{1}{2}\pi-\alpha).$$

In these equations we can, if we please, change the sign of α, since α may be either positive or negative; we have then the equations

$$\left.\begin{aligned}\sin(\tfrac{1}{2}\pi\pm\alpha)&=\cos\alpha\\ \cos(\tfrac{1}{2}\pi\pm\alpha)&=\mp\sin\alpha\\ \tan(\tfrac{1}{2}\pi\pm\alpha)&=\mp\cot\alpha\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

We have also, from (6) and (9),

$$\begin{aligned}\sin(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=(-1)^m\sin(\tfrac{1}{2}\pi\pm\alpha),\\ \cos(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=(-1)^m\cos(\tfrac{1}{2}\pi\pm\alpha),\\ \tan(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=\tan(\tfrac{1}{2}\pi\pm\alpha),\end{aligned}$$

hence

$$\left.\begin{aligned}\sin(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=(-1)^m\cos\alpha\\ \cos(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=\mp(-1)^m\sin\alpha\\ \tan(\overline{m+\tfrac{1}{2}}\,\pi\pm\alpha)&=\mp\cot\alpha\end{aligned}\right\}\quad\ldots\ldots\ldots(11).$$

The angle $\pi-\alpha$ is called the *supplement* of the angle α, and the angle $\frac{1}{2}\pi-\alpha$ is called the *complement* of α.

We have shewn that *the sine of an angle is equal to the sine of the supplementary angle*, and *the cosine of an angle is equal, with opposite sign, to the cosine of its supplement;* also that *the sine of an angle is equal to the cosine of its complement*, and *the cosine of an angle is equal to the sine of its complement.*

The formulae (6) to (11) enable us to find the circular functions of an angle, when we know the values of the circular functions of that angle between zero and $\frac{1}{2}\pi$, which differs from the given angle by a multiple of $\frac{1}{2}\pi$, or also when we know the circular functions of the complement of this latter angle.

Periodicity of the circular functions.

30. When a function $f(x)$ of a variable has the property $f(x)=f(x+k)$ for every value of x, k being a constant, the function $f(x)$ is called periodic; if moreover the quantity k is the least constant for which the function has this property, then k is called the period of the function.

It follows at once that if $f(x)=f(x+k)$, then $f(x)=f(x+nk)$, where n is any positive or negative integer; if then we know the values of the function for all values of x lying between two values of x which differ by k, we know the values of the function for all other values of x, the function having values which are a mere repetition of its values in the interval for which they are given.

The property (6), of $\sin\alpha$ and $\cos\alpha$, shews that these functions are periodic functions of α, the period being 2π, or if the angle is measured in degrees, $\sin A$ and $\cos A$ are periodic functions of A, the period being 360°. The property (7) shews that these functions are such that their values, for values of the angle differing by half the complete period, are equal with opposite sign. The property (8) shews that the tangent is periodic, the complete period being π, half the period of the sine and cosine. Obviously the period of the secant or of the cosecant is 2π, and that of the cotangent is π. It will be hereafter seen that the circular functions derive their importance in analysis principally from their possession of this property of periodicity.

Changes in the sign and magnitude of the circular functions.

31. We shall now trace the changes in the magnitude and sign of the circular functions of an angle, as the angle increases from zero to four right angles.

(1) To trace the changes in the value of the sine of an angle,

we must observe the changes in magnitude and sign of the projection ON, in the figure of Art. 18. When the angle A is zero, ON is zero, and as A increases up to 90°, ON is positive and increases until when A is 90°, ON is equal to OP, thus $\sin A$ is positive and increases from 0 to 1. As A increases from 90° to 180°, ON is positive and diminishes until when A is 180° it is again zero, therefore $\sin A$ is positive and decreases from 1 to 0. As A increases from 180° to 270°, ON is negative and increases numerically, until when A is 270°, $ON = -OP$, hence $\sin A$ is negative and changes from 0 to -1. As A increases from 270° to 360°, ON is negative and diminishes numerically, until when A is 360° it is again zero, thus $\sin A$ is negative and changes from -1 to 0.

(2) In the case of the cosine, we must observe the changes in magnitude and sign of the projection OM. We find that as A increases from 0° to 90°, $\cos A$ is positive and diminishes from 1 to 0; as A increases from 90° to 180°, $\cos A$ is negative and changes from 0 to -1; as A increases from 180° to 270°, $\cos A$ is negative and changes from -1 to 0; and as A increases from 270° to 360°. $\cos A$ is positive and increases from 0 to 1.

(3) To trace the changes in the tangent of an angle, we must consider the ratio of ON to OM; when the angle is zero, this ratio is zero, and is positive and increasing as the angle increases from 0° to 90°; when the angle is 90°, the projection OM is zero, and ON is unity, hence $\tan 90° = \infty$; as A increases from 90° to 180°, the tangent is negative and changes from $-\infty$ to 0. As A increases from 180° to 270°, $\tan A$ is positive, since ON and OM are both negative, and it increases until it again becomes infinite when $A = 270°$. As A increases from 270° to 360°, the tangent is negative and changes from $-\infty$ to 0. It will be observed that $\tan A$ changes from $+\infty$ to $-\infty$ in passing through the value 90°, and from $-\infty$ to $+\infty$ in passing through 270°; to explain this, it is only necessary to remark that as a variable x changes sign by passing through the value zero, its reciprocal $1/x$ changes sign in passing through the value ∞.

(4) The changes in the values of the cosecant, secant, and cotangent of A may be deduced from the above, if we remember that they are the reciprocals of the sine, cosine, and tangent, respectively. Their values for $A = 0°$, 90°, 180°, 270°, 360° are

given in the following table, which also includes the results obtained above for the sine, cosine, and tangent.

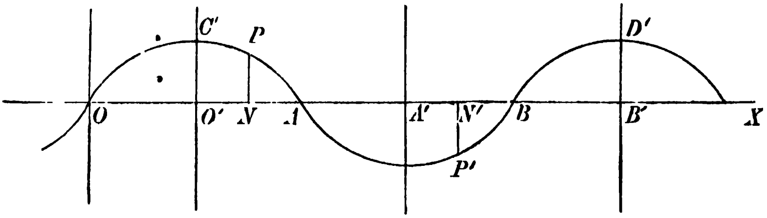
	0°	0°–90°	90°	90°–180°	180°	180°–270°	270°	270°–360°	360°
sin	0	+	1	+	0	−	-1	−	0
cos	1	+	0	−	-1	−	0	+	1
tan	0	+	$\pm\infty$	−	0	+	$\pm\infty$	−	0
cot	$\mp\infty$	+	0	−	$\mp\infty$	+	0	−	$\mp\infty$
sec	1	+	$\pm\infty$	−	-1	−	$\mp\infty$	+	1
cosec	$\mp\infty$	+	1	+	$\pm\infty$	−	-1	−	$\mp\infty$

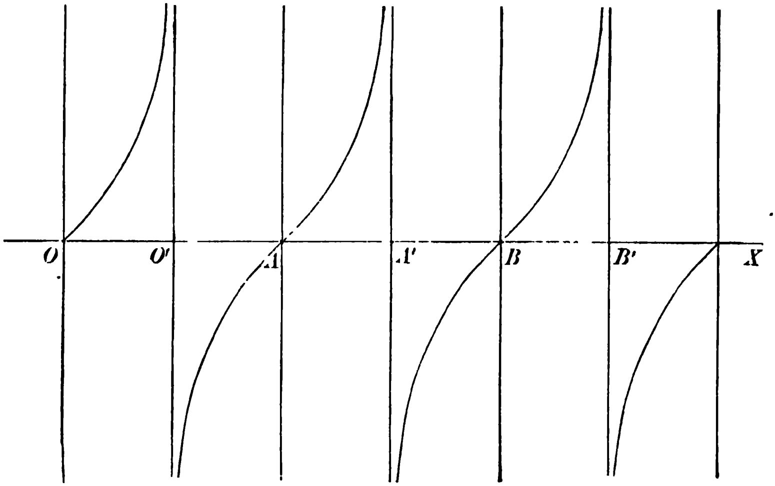
Graphical representation of the circular functions.

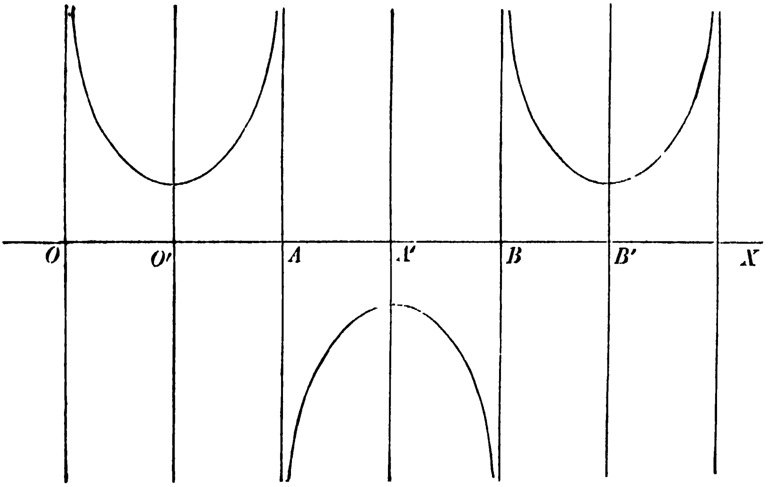
32. In order to obtain a graphical representation of the changes in value of the circular functions, we shall suppose that the circular measure x of an angle is represented by taking a length x measured along a fixed straight line, according to any fixed scale, from a fixed point, and that the numerical value of the function to be represented is the length of a corresponding ordinate drawn perpendicularly to the given straight line, through the extremity of the length x; the function is represented graphically by the curve traced out by the extremity of this ordinate. This curve is called the *graph* of the function.

The first of the three figures opposite gives the graphs of $\sin x$ and of $\cos x$. If O is the origin from which the length x is measured along the fixed straight line OX, and $OA=\pi$, $OB=2\pi$, $OO'=\frac{1}{2}\pi$, $O'C'=1$, the curve $OPAP'B$ is such that any ordinate represents roughly the value of $\sin x$ corresponding to any value of x between 0 and 2π. If O' is taken as origin, and $O'B'=2\pi$, the curve $C'PP'D'$ represents the value of $\cos x$ for values of x between 0 and 2π; this follows from the relation $\cos x=\sin(\frac{1}{2}\pi+x)$. Beyond OB, the curve $OPP'B$ will be repeated indefinitely on both sides of the origin O. The second figure represents, in a similar manner, the values of $\tan x$ and $\cot x$, O being the origin for $\tan x$, and O' for $\cot x$; the ordinates through O', A', B' are asymptotes of the curve, where the functions change sign by passing through an infinite value. The third figure represents the values of $\sec x$

and cosec x, O being the origin for cosec x, and O' for sec x; the ordinates at O, A, B are asymptotes of this curve.







Example. *Draw graphs of the following functions*

(1) $\sin x + \cos x$.

(2) $\cos(\pi \sin x) . \cos(\pi \cos x)$.

(3) $\tan x + \sec x$.

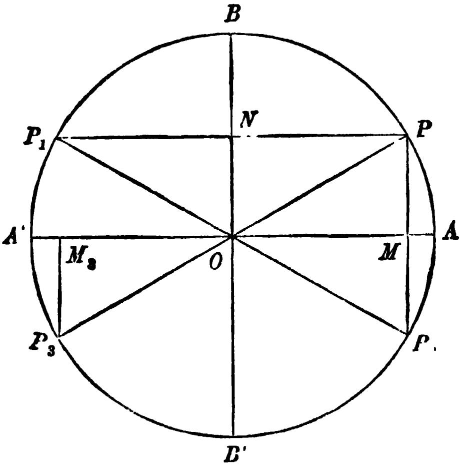
(4) $\sin(\pi \cos x)/\cos(\pi \sin x)$.

(5) $\sin^2 x - 2 \cos x$.

(6) $\sin(\frac{1}{6}\pi + \frac{1}{3}\pi \cos x)$.

Angles with one circular function the same.

33. We shall now find expressions for all the angles which have one of their circular functions the same.



(1) If in the figure, AOP is a given angle, and PP_1 is drawn parallel to OA, the angles (OA, OP) and (OA, OP_1) are the only angles which have their sine the same as that of AOP, for they are the only angles for which the projection of the radius on OB is equal to ON; these angles are $2n\pi + \alpha$ and $2n\pi + \pi - \alpha$, where α is the circular measure of AOP, and n is any integer; they are both included in the expression $m\pi + (-1)^m \alpha$, where m is any positive or negative integer; this is therefore the expression for all the angles whose sine is the same as that of α.

(2) Next draw PP_2 parallel to OB, then the angles (OA, OP) and (OA, OP_2) are the only angles which have the same cosine as α, for they are the only angles for which the projection of OP on OA is equal to OM; they are both included in the formula $2m\pi \pm \alpha$, where m is any positive or negative integer.

(3). If PO is produced to P_3, the angles (OA, OP), (OA, OP_3) are the only ones which have the same tangent as α; these angles are respectively $2n\pi + \alpha$ and $2n\pi + \pi + \alpha$, and are therefore both included in the formula $m\pi + \alpha$, where m is any positive or negative integer.

(4) Since angles which have the same cosecant have also the same sine, we see that $m\pi + (-1)^m \alpha$ includes all the angles whose cosecant is the same as that of α; also $2m\pi \pm \alpha$ includes all angles whose secant is the same as that of α, and $m\pi + \alpha$ includes all angles whose cotangent is the same as that of α.

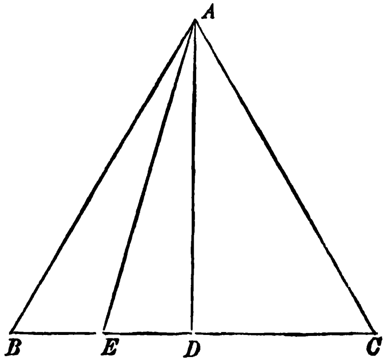
In every case zero is included as one value of m or n.

Determination of the circular functions of certain angles.

34. The values of the circular functions of a few important angles can be obtained by simple geometrical means.

(1) The angle 45° or $\frac{1}{4}\pi$ is an acute angle in a right-angled isosceles triangle, the sine and cosine of this angle are therefore obviously equal to one another; and since the sum of their squares is unity, each of them is equal to $1/\sqrt{2}$; the tangent of the angle is therefore unity.

(2) Each of the angles of an equilateral triangle is 60° or $\frac{1}{3}\pi$.



Let ABC be such a triangle; draw AD perpendicular to BC, then the cosine of the angle B is $\frac{BD}{AB}$, and this is equal to $\frac{1}{2}$; the sine of the same angle is $\sqrt{1-\frac{1}{4}} = \frac{1}{2}\sqrt{3}$. The complement of 60°

is 30° or $\frac{1}{6}\pi$, hence we have $\cos 30^\circ = \frac{1}{2}\sqrt{3}$, and $\sin 30^\circ = \frac{1}{2}$. We have also $\tan 60^\circ = \sqrt{3}$, and $\tan 30^\circ = 1/\sqrt{3}$.

(3) Draw AE bisecting the angle DAB, then the angle DAE is 15° or $\frac{1}{12}\pi$. We have by Euclid, Book VI. Prop. III.

$$\frac{DE}{EB} = \frac{DA}{AB} = \tfrac{1}{2}\sqrt{3},$$

therefore
$$\frac{DE}{DB} = \frac{\sqrt{3}}{2+\sqrt{3}},$$

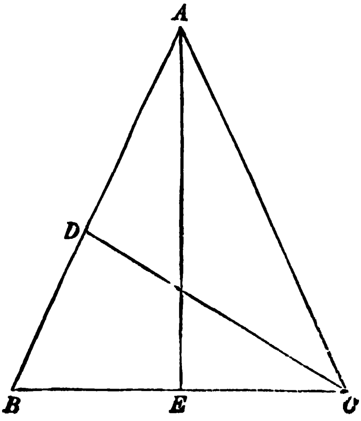
and thence $\dfrac{DE}{DA}$ or $\tan 15^\circ$ is equal to $\dfrac{\sqrt{3}}{\sqrt{3}\,(2+\sqrt{3})}$ or $2-\sqrt{3}$. From this we obtain

$$\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}, \qquad \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

We can, from these values, obtain the sine, cosine, and tangent of 75° or $\frac{5}{12}\pi$, the complementary angle. If we proceeded in the same way, bisecting the angle DAE, we should obtain the tangent of $7^\circ\ 30'$ or $\frac{1}{24}\pi$, and we might continue the process so as to obtain the tangent of all angles of the form $\dfrac{\pi}{3\,.\,2^p}$, where p is a positive integer, but we shall hereafter obtain formulae by which the functions of these angles may be successively calculated, thus obviating the necessity of continuing the geometrical process.

By a similar geometrical method we might obtain the circular functions of the angles of the form $\pi/2^p$.

(4) Let ABC be a triangle in which each of the base angles is double of the vertical angle A; the base angles are each 72°, or



$\frac{2}{5}\pi$, and the vertical angle is 36°, or $\frac{1}{5}\pi$. If AB is divided at D so that $AB.BD = AD^2$, then it is shewn in Euclid, Book IV. Prop. X. that $AD = DC = CB$. Draw AE perpendicular to BC. Denoting the ratio of AD to AB by x, we have $1 - x = x^2$, and solving this quadratic, we find $x = \frac{1}{2}(\pm\sqrt{5} - 1)$; we must take the positive root, hence $\frac{AD}{AB} = \frac{1}{2}(\sqrt{5} - 1)$, thus

$$\cos 72^\circ = \sin 18^\circ = \frac{1}{2}\frac{BC}{AB} = \frac{1}{4}(\sqrt{5} - 1);$$

from this we obtain $\sin 72^\circ = \cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$.

Also $\cos 36^\circ = \frac{1}{2}\frac{AC}{AD}$, since DAC is an isosceles triangle, therefore $\cos 36^\circ = \frac{1}{4}(\sqrt{5} + 1)$, hence $\sin 36^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$.

Since 54° is the complement of 36°, we have therefore the values of sin 54° and cos 54°.

In the following table the values we have obtained are collected for reference. The functions in the first line refer to the angles in the first column, and the functions in the last line to the angles in the last column.

	sine	cosine	tangent	cotangent	
$\frac{1}{12}\pi = 15^\circ$	$\frac{\sqrt{6} - \sqrt{2}}{4}$	$\frac{\sqrt{6} + \sqrt{2}}{4}$	$2 - \sqrt{3}$	$2 + \sqrt{3}$	$\frac{5}{12}\pi = 75^\circ$
$\frac{1}{10}\pi = 18^\circ$	$\frac{\sqrt{5} - 1}{4}$	$\frac{\sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{1}{5}\sqrt{25 - 10\sqrt{5}}$	$\sqrt{5 + 2\sqrt{5}}$	$\frac{2}{5}\pi = 72^\circ$
$\frac{1}{6}\pi = 30^\circ$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{1}{3}\pi = 60^\circ$
$\frac{1}{5}\pi = 36^\circ$	$\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	$\frac{\sqrt{5} + 1}{4}$	$\sqrt{5 - 2\sqrt{5}}$	$\frac{1}{5}\sqrt{25 + 10\sqrt{5}}$	$\frac{3}{10}\pi = 54^\circ$
$\frac{1}{4}\pi = 45^\circ$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\frac{1}{4}\pi = 45^\circ$
	cosine	sine	cotangent	tangent	

We can find at once the circular functions of any angle which differs from any one of those in the table by a multiple of a right angle, by employing the formulae (6) to (11).

EXAMPLE. *Find the sine and cosine of* 120°, *and of* −576°.

We have $120° = 90° + 30°$, hence

$$\sin 120° = \cos 30° = \tfrac{1}{2}\sqrt{3}, \quad \cos 120° = -\sin 30° = -\tfrac{1}{2}.$$

Again $-576° = -(3 \,.\, 180° + 36°)$, therefore

$$\sin(-576°) = \sin(+180° - 36°) = \sin 36°,$$

also $\cos -576° = \cos(180° - 36°) = -\cos 36°$.

The inverse circular functions.

35. If y is a function $f(x)$ of x, then x may also be regarded as a function of y; this function of y is called the inverse function of $f(x)$, and is usually denoted by $f^{-1}(y)$; thus $x = f^{-1}(y)$. If $f(x)$ is a periodic function, of period k, so that $f(x) = f(x + mk)$, where m is any positive or negative integer, the function $f^{-1}(y)$ will have an infinite number of values given by $x + mk$, where x is any one value of $f^{-1}(y)$; such a function of y is called *multiple-valued*, since it has not a single value for each value of the variable y. We see therefore that, *corresponding to a periodic function* f(x) = y, *there is a multiple-valued inverse function* $\mathrm{f}^{-1}(\mathrm{y})$ *which has an infinite number of values for any one value of* y, *these values differing by multiples of the period of* f(x).

36. If there are two or more values of x, lying between 0 and k, for which $f(x)$ has equal values, the multiplicity of values of $f^{-1}(y)$ is still further increased, since it will have each of the values of x for which $f(x) = y$, and the infinite series of values obtained by adding multiples of k to each of these. For example, suppose that there are two values x_1, x_2, each lying between 0 and k, for which $f(x) = y$, then the inverse function $f^{-1}(y)$ has the two sets of values $x_1 + nk$, $x_2 + nk$.

37. In the case of the circular function $\sin x = y$, the values of the inverse function $\sin^{-1} y$ are $n\pi + (-1)^n x_1$, where x_1 is any value of x for which $\sin x_1 = y$; in this case the complete period of $\sin x$ is 2π, and there are two values of x, say x_1 and $\pi - x_1$, lying between 0 and 2π, for which $\sin x = y$; thus the values of $\sin^{-1} y$ are the two series of values $n.2\pi + x_1$ and $n.2\pi + \pi - x_1$, both included in $n\pi + (-1)^n x_1$.

In a similar manner, we see that the values of $\cos^{-1} y$ are included in $2n\pi \pm x$, where $\cos x = y$.

The periods of the functions $\tan x$, $\cot x$ are π, only half those of $\sin x$ and $\cos x$, and there is only one value of x between 0 and π for which $\tan x$ or $\cot x$ has any given value; thus $\tan^{-1} y$ has the values $n\pi + x_1$, and $\cot^{-1} y$ the values $n\pi + x_1$, where x_1 is that value of x between 0 and π, such that $\tan x_1$ or $\cot x_1$ is equal to y.

38. The numerically smallest quantity x which has the same sign as y, and is such that $\sin x = y$, is called the *Principal Value* of $\sin^{-1} y$; a similar definition applies to the principal values of $\tan^{-1} y$, $\cot^{-1} y$, $\operatorname{cosec}^{-1} y$.

The numerically smallest positive value of x which is such that $\cos x = y$ is called the Principal Value of $\cos^{-1} y$; a similar definition applies to $\sec^{-1} y$.

Thus the principal values of $\sin^{-1} y$, $\tan^{-1} y$, $\cot^{-1} y$, $\operatorname{cosec}^{-1} y$ lie between the values $\pm \frac{1}{2}\pi$, and the principal values of $\cos^{-1} y$, $\sec^{-1} y$ lie between 0 and π. In some works, the principal values of $\sin^{-1} y$, $\cos^{-1} y$, $\tan^{-1} y$ are denoted by $\mathrm{Sin}^{-1} y$, $\mathrm{Cos}^{-1} y$, $\mathrm{Tan}^{-1} y$; the general values are then given by

$$\sin^{-1} y = n\pi + (-1)^n \mathrm{Sin}^{-1} y, \cos^{-1} y = 2n\pi \pm \mathrm{Cos}^{-1} y, \tan^{-1} y = n\pi + \mathrm{Tan}^{-1} y;$$

we shall however not use this notation. It must be remembered that in many equations connecting these inverse functions it is necessary to suppose that the functions have their principal values, or at all events that the choice of values is restricted. For example, in such an equation as $\sin^{-1} y + \cos^{-1} y = \frac{1}{2}\pi$, the choice of values of the inverse functions is restricted.

It should moreover be noticed that the functions $\cos^{-1} y$, $\sin^{-1} y$ have only been defined for values of y lying between ± 1; beyond those limits of y, the functions have no meaning, so far as they have been at present defined. The student should draw, as an exercise, graphs of the various inverse circular functions.

In Continental works, the notation arc $\sin x$, arc $\cos x$, arc $\tan x$ is used for $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

EXAMPLES ON CHAPTER III.

1. Prove the identities

   (i) $\tan A (1 - \cot^2 A) + \cot A (1 - \tan^2 A) = 0$,

   (ii) $(\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2$.

2. The sine of an angle is $\dfrac{m^2 - n^2}{m^2 + n^2}$; find the other circular functions.

3. If $$\tan A + \sin A = m, \quad \tan A - \sin A = n,$$
prove that $$m^2 - n^2 = 4\sqrt{mn}.$$

4. Having given $\frac{\sin A}{\sin B} = p$, $\frac{\cos A}{\cos B} = q$, find $\tan A$ and $\tan B$.

5. If $$\frac{\sin A}{\sin B} = \sqrt{2}, \quad \frac{\tan A}{\tan B} = \sqrt{3}, \quad \text{find } A \text{ and } B.$$

6. If $$\cos A = \tan B, \quad \cos B = \tan C, \quad \cos C = \tan A,$$
prove that $$\sin A = \sin B = \sin C = 2 \sin 18^\circ$$

7. Solve the equations:

$$\text{(i)} \quad \sin\theta + 2\cos\theta = 1,$$

$$\text{(ii)} \quad \frac{\cos a}{\tan a} = \frac{3}{2},$$

$$\text{(iii)} \quad \sqrt{3}\operatorname{cosec}^2\theta = 4\cot\theta.$$

8. Solve the equations:

$$\left.\begin{aligned}\cos(2x+y) &= \sin(x-2y)\\ \cos(x+2y) &= \sin(2x-y)\end{aligned}\right\}.$$

9. Find a general expression for θ, when $\sin^2\theta = \sin^2 a$, and also when
$$\sin\theta = -\cos\theta = 1/\sqrt{2}.$$

10. Find the general values of the limits between which A lies, when $\sin^2 A$ is greater than $\cos^2 A$.

11. Find the general value of θ, when $9\sec^4\theta = 16$.

12. If $$\tan(\pi\cot\theta) = \cot(\pi\tan\theta),$$
then $$\tan\theta = \tfrac{1}{4}\{2n + 1 \pm \sqrt{4n^2 + 4n - 15}\},$$
where n is any integer which does not lie between 1 and -2.

13. Give geometrical constructions for dividing a given angle into two parts, so that (1) the sines, (2) the tangents of the two parts may be in a given ratio.

14. Construct the angle whose tangent is $3 - \sqrt{2}$.

15. Divide a given angle into two parts the sum of whose cosines may be a given quantity c. What are the greatest and least values c can have?

16. If $$u_n = \cos^n\theta + \sin^n\theta,$$
prove that $$2u_6 - 3u_4 + 1 = 0,$$
$$6u_{10} - 15u_8 + 10u_6 - 1 = 0.$$

17. Two circles of radii a, b touch each other externally; θ is the angle contained by the common tangents to these circles, prove that

$$\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

18. A pyramid has for base a square of side a; its vertex lies on a line through the middle point of the base, perpendicular to it, and at a distance h from it; prove that the angle α between two lateral faces is given by

$$\sin\alpha=\frac{2h\sqrt{2a^2+4h^2}}{a^2+4h^2}.$$

19. Two planes intersect at right angles in a line AB, and a third plane cuts them in lines AD, AC; if the angles DAB, CAB be denoted by α, β respectively, prove that the angle BA makes with the plane CAD is

$$\tan^{-1}\frac{\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}}.$$

20. Shew that, if OD be the diagonal of a rectangular parallelepiped, the cosines of the angles between OD and the diagonals of the face of which OA, OB are adjacent sides are respectively

$$\frac{AB}{OD} \text{ and } \frac{OA^2\sim OB^2}{OD.AB}.$$

21. Two circles, the sum of whose radii is a, are placed in the same plane, with their centres at a distance $2a$, and an endless string, quite stretched, partly surrounds the circles, and crosses itself between them. Shew that the length of the string is $(\frac{4}{3}\pi+2\sqrt{3})\,a$.

22. Prove that

$$\cos\tan^{-1}\sin\cot^{-1}x=\left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}.$$

23. Illustrate graphically the change in sign and magnitude of the functions $3\sin x+4\cos x$, $e^x\sin x$, and $\sin\left(\frac{\pi}{\sqrt{2}}\sin x\right)$ for all values of x.

Shew that the equation $2x=(2n+1)\,\pi\operatorname{vers}x$, where n is a positive integer, has $2n+3$ real roots and no more, roughly indicating their localities.

CHAPTER IV.

THE CIRCULAR FUNCTIONS OF TWO OR MORE ANGLES.

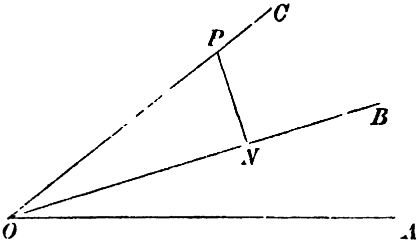
The addition and subtraction formulae for the sine and cosine.

39. WE shall now find expressions for the circular functions of the sum and of the difference of two angles, in terms of the circular functions of those angles.

Suppose an angle AOB of any magnitude A, positive or negative, to be generated by a straight line revolving round O from the initial position OA, our usual convention being made as to the sign of the angle, and suppose further that an angle BOC of any magnitude B is described by a line revolving from the initial position OB; then the angle AOC is equal to $A+B$. In OC take a point P, and draw PN perpendicular to OB.

According to the convention in Art. 15, the straight line ON is positive or negative according as it is in OB, or in OB produced; also NP is positive when it is on the positive side of OB, revolving counter-clockwise, and negative when on the other side. The positive direction of the straight line on which NP lies makes an angle $A+90°$ with OA. We have $ON=OP\cos B$, and $NP=OP\sin B$; for ON and NP are the projections of OP on OB and on the line which makes an angle $A+90°$ with OA.

In fig. (1), each of the angles A, B is positive and less than 90°; in fig. (2), the angle A lies between 90° and 180°, and the angle B also lies between 90° and 180°; in fig. (3), the angle A lies between 180° and 270°, and the angle B is negative and lies between $-90°$ and $-180°$. In figs. (1) and (2), NP is of positive length, and in fig. (3), NP is of negative length, since, in the last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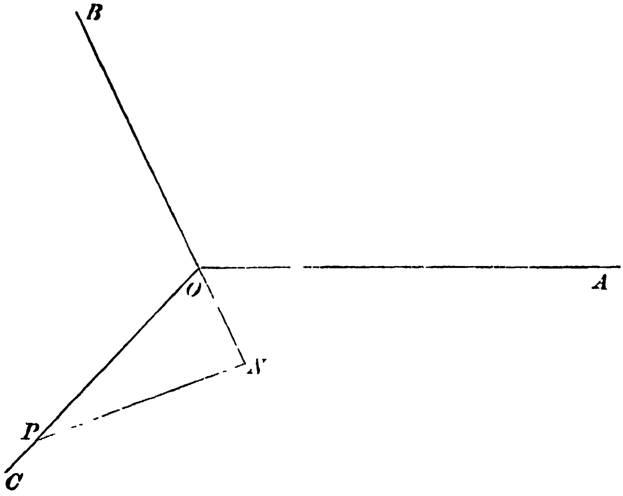
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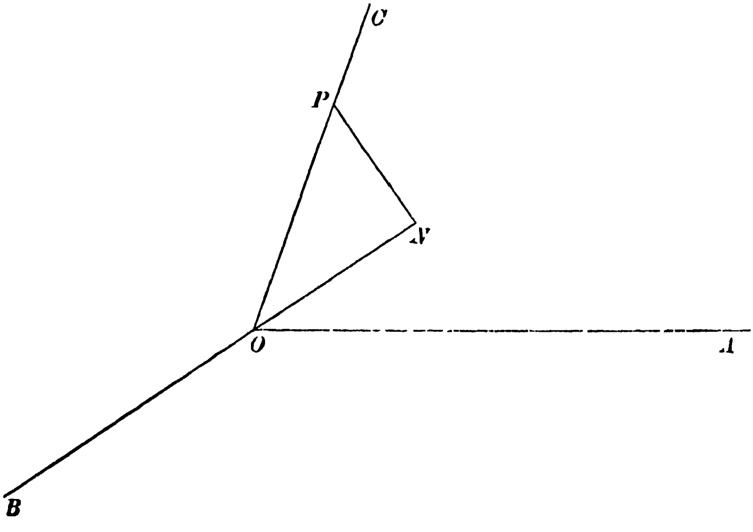
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case, PN is the direction of a line making an angle $A+90°$ with OA.

By the fundamental theorem in projections, given in Art. 17, the projection of OP on OA is equal to the sum of the projections of ON and NP on OA, or

$$\begin{aligned} OP\cos(A+B) &= ON\cos A + NP\cos(A+90°) \\ &= OP\cos A\cos B + OP\sin B\cos(A+90°), \end{aligned}$$

therefore $\cos(A+B)=\cos A\cos B-\sin A\sin B$...........(1).

If, instead of projecting the sides of the triangle ONP on OA, we project them on a line making an angle $+90°$ with OA, we have

$$\begin{aligned} OP\sin(A+B) &= ON\sin A + NP\sin(A+90°) \\ &= OP\sin A\cos B + OP\sin(A+90°)\sin B, \end{aligned}$$

therefore $\sin(A+B)=\sin A\cos B+\cos A\sin B$...........(2).

The formulae (1) and (2) have thus been proved for angles of all magnitudes, both positive and negative. The student should draw the figure, for various magnitudes of the angles A and B, in order to convince himself of the generality of the proof.

If we change B into $-B$, in each of the formulae (1) and (2), we have

$\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)$

and $\sin(A-B)=\sin A\cos(-B)-\cos A\sin(-B)$,

hence $\cos(A-B)=\cos A\cos B+\sin A\sin B$ ....(3),

and $\sin(A-B)=\sin A\cos B-\cos A\sin B$ ....(4).

These formulae (3) and (4) would of course be obtained directly, by describing the angle B in the figure in the negative direction, so that the angle POA would be equal to $A-B$.

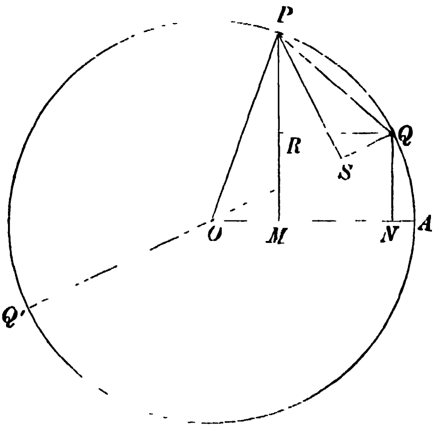
40. The formulae (1), (2), and (3), (4), are called the addition and subtraction formulae respectively; either of the formulae (1) and (2) may be at once deduced from the other; in (1) write $A+90°$ for A, we have then

$$\cos(90°+A+B)=\cos(90°+A)\cos B-\sin(90°+A)\sin B$$

or $$-\sin(A+B)=-\sin A\cos B-\cos A\sin B;$$

and changing the signs on both sides of this equation, we have the formula (2); in the same way, by writing $A+90°$ for A in (2), we should obtain (1). It appears then that all these four fundamental formulae are really contained in any one of them.

41. The proof of the addition and subtraction formulae, given by Cauchy, is as follows:—With O as centre describe a circle, and let the radii OP, OQ



make angles A, B, respectively, with OA; join PQ, and draw PM, QN perpendicular to OA, and QR parallel to OA, then we have

$$\begin{aligned} PQ^2 &= QR^2 + RP^2 \\ &= (ON - OM)^2 + (PM - QN)^2 \\ &= OA^2\{(\cos B - \cos A)^2 + (\sin A - \sin B)^2\} \\ &= 2OA^2(1 - \cos A\cos B - \sin A\sin B). \end{aligned}$$

Let PS be drawn perpendicular to the diameter QQ', then

$$\begin{aligned} PQ^2 = QS \,.\, QQ' &= 2OA\,(OA - OS) \\ &= 2OA^2\{1 - \cos(A - B)\}, \end{aligned}$$

therefore $$\cos(A - B) = \cos A\cos B + \sin A\sin B \text{..................} (3).$$

The other formulae may then be deduced; (1) by changing B into $-B$, (2) by changing B into $90° - B$, (4) by changing B into $90° + B$.

42. Besides the two proofs which we have given of the fundamental addition and subtraction formulae, both of which are perfectly general, various other proofs have been given, some of which are in the first instance only applicable to angles between a limited range of values, and require extension in the cases of angles whose magnitudes are beyond that range. We shall make this extension in the case in which the formulae have been first proved for values of A and B between 0° and 90°. Whatever A and B are, it is always possible to find angles A' and B', lying between 0° and 90°, such that $A = m \,.\, 90° + A'$, $B = n \,.\, 90° + B'$, where m and n are positive or negative integers; we have then

$$\cos (A + B) = \cos (\overline{m+n}\, 90° + A' + B');$$

(1) if m and n are both even, we have

$$\begin{aligned}\cos (A + B) &= (-1)^{\frac{m+n}{2}} \cos (A' + B') \\ &= (-1)^{\frac{m+n}{2}} (\cos A' \cos B' - \sin A' \sin B'),\end{aligned}$$

now $$\cos A = (-1)^{\frac{m}{2}} \cos A', \quad \sin A = (-1)^{\frac{m}{2}} \sin A',$$

with similar formulae for B,

hence $$\cos (A + B) = \cos A \cos B - \sin A \sin B;$$

(2) if m and n are both odd, we have

$$\begin{aligned}\cos A &= (-1)^{\frac{m-1}{2}} \cos (90° + A') = (-1)^{\frac{m+1}{2}} \sin A', \\ \sin A &= (-1)^{\frac{m-1}{2}} \sin (90° + A') = (-1)^{\frac{m-1}{2}} \cos A',\end{aligned}$$

with similar formulae for B; hence as before we obtain, by substituting the values of $\cos A'$, $\cos B'$, $\sin A'$, $\sin B'$, the formula for $\cos (A + B)$;

(3) if m is odd and n is even,

$$\begin{aligned}\cos (A + B) &= (-1)^{\frac{m+n-1}{2}} \cos (90° + A' + B') \\ &= (-1)^{\frac{m+n+1}{2}} \sin (A' + B') \\ &= (-1)^{\frac{m+n+1}{2}} (\sin A' \cos B' + \cos A' \sin B'),\end{aligned}$$

now $$\begin{aligned}\cos A &= (-1)^{\frac{m+1}{2}} \sin A', \quad \cos B = (-1)^{\frac{n}{2}} \cos B', \\ \sin A &= (-1)^{\frac{m-1}{2}} \cos A', \quad \sin B = (-1)^{\frac{n}{2}} \sin B';\end{aligned}$$

hence, substituting as before, we have the formula for $\cos (A + B)$. The other formulae may be extended in the same manner.

43. The form in which the addition formulae were known in the Greek Trigonometry[1] is Ptolemy's theorem given in Euclid, Bk. VI. Prop. D; this theorem is, that if $ABCD$ be a quadrilateral inscribed in a circle, $AB \,.\, CD + AD \,.\, BC = AC \,.\, BD$. Any chord AB is the sine of half the angle which AB subtends at the centre of the circle, the diameter of the circle being taken as unity, and this half angle is the angle subtended by the arc AB at the circumference. We shall shew that the formulae for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ are contained in Ptolemy's theorem.

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[1] See the Article "Ptolemy" in the *Encyclopaedia Britannica*, ninth Edition.

(1) Let BD be a diameter of the circle, and $ADB = \alpha$, $BDC = \beta$; then $ABD = \frac{1}{2}\pi - \alpha$, $DBC = \frac{1}{2}\pi - \beta$, $AC = \sin(\alpha + \beta)$, $AB = \sin\alpha$, $CD = \cos\beta$; thus the theorem is equivalent to the formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

(2) Let CD be a diameter of the circle, and $BCD = \alpha$, $ACD = \beta$, thus $AB = \sin(\alpha - \beta)$, and the theorem is equivalent to

$$\sin(\alpha - \beta) + \sin\beta\cos\alpha = \cos\beta\sin\alpha.$$

(3) Let BD be a diameter of the circle, and $ADB = \alpha$, $CBD = \beta$, then $ADC = \frac{1}{2}\pi + \alpha - \beta$, thus $AC = \cos(\alpha - \beta)$, and the theorem is equivalent to

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(4) Let CD be a diameter of the circle, and $BCD = \alpha$, $ADC = \beta$; then $BCA = \alpha + \beta - \frac{1}{2}\pi$, $AB = -\cos(\alpha + \beta)$, and the theorem is equivalent to

$$-\cos(\alpha + \beta) + \cos\alpha\cos\beta = \sin\alpha\sin\beta.$$

EXAMPLE. *Employ Ptolemy's theorem to prove the following theorems:*

$$\sin\alpha\sin(\beta - \gamma) + \sin\beta\sin(\gamma - \alpha) + \sin\gamma\sin(\alpha - \beta) = 0,$$
$$\sin(\alpha + \beta)\sin(\beta + \gamma) = \sin\alpha\sin\gamma + \sin\beta\sin(\alpha + \beta + \gamma).$$

Formulae for the addition or subtraction of two sines or two cosines.

44. We obtain at once from the addition and subtraction formulae

$$\sin(A + B) + \sin(A - B) = 2\sin A\cos B,$$
$$\sin(A + B) - \sin(A - B) = 2\cos A\sin B,$$
$$\cos(A + B) + \cos(A - B) = 2\cos A\cos B,$$
$$\cos(A - B) - \cos(A + B) = 2\sin A\sin B,$$

let $A + B = C$, $A - B = D$; we obtain then, since $A = \frac{1}{2}(C + D)$, $B = \frac{1}{2}(C - D)$, the formulae

$$\sin C + \sin D = 2\sin\tfrac{1}{2}(C + D)\cos\tfrac{1}{2}(C - D) \quad \text{......(5)},$$
$$\sin C - \sin D = 2\cos\tfrac{1}{2}(C + D)\sin\tfrac{1}{2}(C - D) \quad \text{......(6)},$$
$$\cos C + \cos D = 2\cos\tfrac{1}{2}(C + D)\cos\tfrac{1}{2}(C - D) \quad \text{......(7)},$$
$$\cos D - \cos C = 2\sin\tfrac{1}{2}(C + D)\sin\tfrac{1}{2}(C - D) \quad \text{......(8)}.$$

These important formulae (5), (6), (7), (8) are the expressions for the sum or difference of the sines or of the cosines of two angles as products of two circular functions; they may be expressed in words as follows:

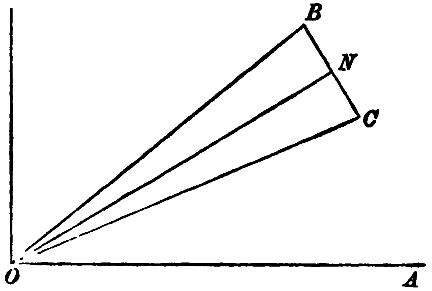
The sum of the sines of two angles is equal to twice the product of the sine of half the sum and the cosine of half the difference of the angles.

The difference of the sines of two angles is equal to twice the product of the cosine of half the sum and the sine of half the difference of the angles.

The sum of the cosines of two angles is equal to twice the product of the cosine of half the sum and the cosine of half the difference of the angles.

The difference of the cosines of two angles is equal to twice the product of the sine of half the sum and the sine of half the reversed difference of the angles.

45. These formulae may be proved geometrically by the method of projections.



Let $BOA = C$, $COA = D$, and let $OB = OC$; draw ON perpendicular to BC, then N is the middle point of BC, also

$$NOA = \tfrac{1}{2}(C + D), \quad NOB = NOC = \tfrac{1}{2}(C - D).$$

The sum of the projections of OB, OC, on OA, is equal to the sum of the projections of ON, NB, ON, NC, on OA, and, since the projections of NB and NC are equal with opposite sign, this is equal to twice the projection of ON; therefore

$$OB \cos C + OC \cos D = 2ON \cos \tfrac{1}{2}(C + D),$$

and since $$ON = OB \cos \tfrac{1}{2}(C - D),$$

we have the formula

$$\cos C + \cos D = 2 \cos \tfrac{1}{2}(C + D) \cos \tfrac{1}{2}(C - D) \ldots\ldots\ldots (7).$$

If instead of projecting on OA we project on a straight line perpendicular to OA, we have

$$OB \sin C + OC \sin D = 2ON \sin \tfrac{1}{2}(C + D),$$

hence $\sin C + \sin D = 2 \sin \tfrac{1}{2}(C + D) \cos \tfrac{1}{2}(C - D) \ldots\ldots\ldots (5)$.

Also the projection of OC on OA is equal to the projection of OB, together with twice the projection of BN, or

$$OC \cos D = OB \cos C + 2BN \sin \tfrac{1}{2}(C + D),$$

hence $\cos D - \cos C = 2 \sin \tfrac{1}{2}(C + D) \sin \tfrac{1}{2}(C - D) \ldots\ldots\ldots (8)$,

and if we project on the line perpendicular to OA, we have

$$OC \sin D = OB \sin C - 2BN \cos \tfrac{1}{2}(C + D)$$

or $\sin C - \sin D = 2 \sin \tfrac{1}{2}(C - D) \cos \tfrac{1}{2}(C + D) \ldots\ldots\ldots (6)$.

A curious method of multiplying numbers, by means of tables of sines, was in use for about a century before the invention of logarithms. This method depended on a use of the formula

$$\sin A \sin B = \tfrac{1}{2}\{\cos(A - B) - \cos(A + B)\};$$

the angles A and B, whose sines, omitting the decimal point, are equal to the numbers to be multiplied, can be found from a table of sines, and then $\cos(A + B)$, $\cos(A - B)$ can be found from the same table; half the difference of these last gives the required product. This method was called *προσθαφαίρεσις*. An account of this method will be found in a paper by Glaisher, in the *Philosophical Magazine* for 1878, entitled "On Multiplication by a Table of single Entry."

EXAMPLES.

(1) *Prove the identity*

$$\sin A \sin(B - C) \sin(B + C - A) + \sin B \sin(C - A) \sin(C + A - B)$$
$$+ \sin C \sin(A - B) \sin(A + B - C) = 2 \sin(B - C) \sin(C - A) \sin(A - B).$$

The second and third terms on the left-hand side may be written

$$\tfrac{1}{2}\sin B\{\cos(B - 2A) - \cos(2C - B)\} + \tfrac{1}{2}\sin C\{\cos(C - 2B) - \cos(2A - C)\},$$

which is equal to

$$\tfrac{1}{4}\{\sin 2(B - A) + \sin 2A - \sin 2C - \sin 2(B - C)\}$$
$$+ \tfrac{1}{4}\{\sin 2(C - B) + \sin 2B - \sin 2A - \sin 2(C - A)\},$$

or $\tfrac{1}{4}(\sin 2B - \sin 2C) - \tfrac{1}{2}\sin 2(B - C) + \tfrac{1}{4}\{\sin 2(B - A) - \sin 2(C - A)\}$,

or $\sin(B - C)\{\tfrac{1}{2}\cos(B + C) - \cos(B - C) + \tfrac{1}{2}\cos(B + C - 2A)\}$,

which is equal to $\sin(B - C)\{\cos A \cos(B + C - A) - \cos(B - C)\}$;

adding the term $\sin A \sin(B - C) \sin(B + C - A)$,

we have $\sin(B - C)\{\cos(B + C - 2A) - \cos(B - C)\}$,

or $2 \sin(B - C) \sin(C - A) \sin(A - B)$.

(2) *Prove that*

$$\Sigma \cos A \sin(B-C) \sin(B+C-A) = 2 \sin(B-C) \sin(C-A) \sin(A-B).$$

This may be deduced from Ex. (1), by changing A, B, C into $90° - A$, $90° - B$, $90° - C$ respectively, or may be proved independently as in Ex. (1).

Prove the identities

(3) $\Sigma \sin A \sin(B-C) = 0$, $\Sigma \cos A \sin(B-C) = 0$.

(4) $\Sigma \sin(B+C) \sin(B-C) = 0$, $\Sigma \cos(B+C) \sin(B-C) = 0$.

(5) $\Sigma \sin B \sin C \sin(B-C) = -\sin(B-C) \sin(C-A) \sin(A-B)$,

$\Sigma \cos B \cos C \cos(B-C) = -\sin(B-C) \sin(C-A) \sin(A-B)$.

(6) *Prove that if* $A+B+C=\pi$,

$$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A,$$

and

$$\cos^2 A = 1 - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

A large number of Trigonometrical identities are analogous to similar Algebraical identities[1]. For example, the following algebraical identities correspond to examples (1) to (5),

$\Sigma a(b-c)(b+c-a) = 2(b-c)(c-a)(a-b)$, to (1) and (2),

$\Sigma a(b-c) = 0$, to (3), $\Sigma (b+c)(b-c) = 0$, to (4),

$\Sigma bc(b-c) = -(b-c)(c-a)(a-b)$, to (5).

We shall, in Chap. VII., give the theory of these correspondences.

Addition and subtraction formulae for the tangent and cotangent.

46. From the addition and subtraction formulae we may deduce formulae for the tangent or cotangent of the sum or difference of two angles in terms of the tangents or cotangents of those angles. Thus

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B},$$

hence dividing the numerator and the denominator of the fraction by $\cos A \cos B$,

$$\tan(A \pm B) = \frac{\dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B}}{1 \mp \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}};$$

thus we have the two formulae

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{..............(9)},$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{...........(10)}.$$

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[1] A large number of these correspondences are given by M. Gelin, in *Mathesis*, Vol. II.

In a similar manner we obtain the formulae

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}\quad\ldots\ldots\ldots\ldots(11),$$

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}\quad\ldots\ldots\ldots\ldots(12).$$

The formulae (9). (10), (11), (12) are the addition and subtraction formulae for the tangent and cotangent.

Various formulae.

47. The following formulae may be deduced from the formulae which we have obtained for two angles, and are frequently useful in effecting transformations. The student should verify each of them.

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A\ldots(13),$$

$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A\ldots(14),$$

$$\sin(A+B)\cos(A-B)=\sin A\cos A+\sin B\cos B\quad\ldots\ldots\ldots(15),$$

$$\cos(A+B)\sin(A-B)=\sin A\cos A-\sin B\cos B\quad\ldots\ldots\ldots(16),$$

$$\frac{\sin(A+B)}{\sin(A-B)}=\frac{\tan A+\tan B}{\tan A-\tan B}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

$$\frac{\cos(A+B)}{\cos(A-B)}=\frac{1-\tan A\tan B}{1+\tan A\tan B}\quad\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

$$\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos A\cos B}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

From the formulae for the addition and subtraction of two sines or cosines we obtain at once

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}\quad\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

$$\frac{\sin A\pm\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A\pm B)\quad\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

$$\frac{\sin A\pm\sin B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A\mp B)\quad\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

$$\frac{\cos A+\cos B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B)\ldots(23).$$

EXAMPLES.

(1) *Prove the identity*

$$1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$$
$$=4\sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(-A+B+C)\sin\tfrac{1}{2}(A-B+C)\sin\tfrac{1}{2}(A+B-C).$$

The expression on the left-hand side may be written

$$-\cos^2 A-\cos(B+C)\cos(B-C)+\cos A\,\{\cos(B+C)+\cos(B-C)\},$$

which is equal to $\{\cos A-\cos(B+C)\}\{\cos(B-C)-\cos A\}$;

then, splitting each of these factors into two factors, we obtain the expression on the right-hand side. If $\pm A\pm B\pm C$ is a multiple of 2π, then

$$1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$$

is zero; this result is sometimes useful.

(2) *Prove that*

$$1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C$$
$$=-4\cos\tfrac{1}{2}(A+B+C)\cos\tfrac{1}{2}(-A+B+C)\cos\tfrac{1}{2}(A-B+C)\cos\tfrac{1}{2}(A+B-C).$$

This may be deduced from (1), or proved independently.

(3) *Prove that if* $A+B+C=n\pi$,

$$\sin 2A+\sin 2B+\sin 2C=(-1)^{n-1}4\sin A\sin B\sin C.$$

We have

$$\begin{aligned}\sin 2A+\sin 2B+\sin 2C&=2\sin A\cos A+2\sin(n\pi-A)\cos(B-C)\\&=2\sin A\,\{(-1)^n\cos(B+C)-(-1)^n\cos(B-C)\}\\&=(-1)^{n-1}4\sin A\sin B\sin C.\end{aligned}$$

(4) *Prove that, under the same supposition as in Ex.* (3),

$$1+\cos 2A+\cos 2B+\cos 2C=(-1)^n 4\cos A\cos B\cos C.$$

Prove the identities

(5) $\sin 3A=4\sin A\sin(60^\circ+A)\sin(60^\circ-A)$.

(6) $\cos 3A=4\cos A\cos(60^\circ+A)\cos(60^\circ-A)$.

(7) $\sin A+\sin B+\sin C-\sin(A+B+C)$
$$=4\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+A)\sin\tfrac{1}{2}(A+B).$$

(8) $\cos A+\cos B+\cos C+\cos(A+B+C)$
$$=4\cos\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(C+A)\cos\tfrac{1}{2}(A+B).$$

(9) $\Sigma\sin 2A\sin^2(B+C)-\sin 2A\sin 2B\sin 2C$
$$=2\sin(B+C)\sin(C+A)\sin(A+B).$$

(10) $\Sigma\cos 2A\cos^2(B+C)-\cos 2A\cos 2B\cos 2C$
$$=2\cos(B+C)\cos(C+A)\cos(A+B).$$

(11) $\Sigma\sin^2 A\sin(B+C-A)-2\sin A\sin B\sin C$
$$=\sin(B+C-A)\sin(C+A-B)\sin(A+B-C).$$

(12) $\Sigma\cos^2 A\cos(B+C-A)-2\cos A\cos B\cos C$
$$=\cos(B+C-A)\cos(C+A-B)\cos(A+B-C).$$

(9) and (10) correspond to the algebraical identity

$$\Sigma\, 2a(b+c)^2 - 8abc = 2(b+c)(c+a)(a+b);$$

(11) and (12) to the identity

$$\Sigma\, a^2(b+c-a) - 2abc = (b+c-a)(c+a-b)(a+b-c).$$

Addition formulae for three angles.

48. From the addition formulae (1) and (2) we may deduce formulae for the circular functions of the sum of three angles in terms of functions of those angles; we have

$$\begin{aligned}\sin(A+B+C) &\\ &= \sin(A+B)\cos C + \cos(A+B)\sin C\\ &= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C,\end{aligned}$$

and

$$\begin{aligned}\cos(A+B+C) &\\ &= \cos(A+B)\cos C - \sin(A+B)\sin C\\ &= (\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C,\end{aligned}$$

hence we have

$$\begin{aligned}\sin(A+B+C) &\\ &= \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B\\ &\qquad - \sin A\sin B\sin C \quad\text{......(24)},\end{aligned}$$

$$\begin{aligned}\cos(A+B+C) &\\ &= \cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin C\sin A\\ &\qquad - \cos C\sin A\sin B \quad\text{......(25)}.\end{aligned}$$

The formulae (24), (25) may be written in the form

$$\begin{aligned}\sin(A+B+C) &\\ &= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C),\end{aligned}$$

$$\begin{aligned}\cos(A+B+C) &\\ &= \cos A\cos B\cos C(1 - \tan B\tan C - \tan C\tan A - \tan A\tan B),\end{aligned}$$

hence by division we have the formula

$$\begin{aligned}\tan(A+B+C) &\\ &= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B} \quad\text{......(26)}.\end{aligned}$$

We might obtain in a similar manner the formula

$$\begin{aligned}\cot(A+B+C) &\\ &= \frac{\cot A\cot B\cot C - \cot A - \cot B - \cot C}{\cot B\cot C + \cot C\cot A + \cot A\cot B - 1} \quad\text{......(27)}.\end{aligned}$$

EXAMPLES.

(1) *Prove that* $\tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A$.

(2) *Prove that if* $A+B+C=n\pi$,

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0;$$

and if $A+B+C=(2m+1)\frac{\pi}{2}$,

$$\tan B\tan C+\tan C\tan A+\tan A\tan B=1;$$

and state the corresponding theorems for the cotangents.

Addition formulae for any number of angles.

49. It is obvious that we might now obtain formulae for the circular functions of the sum of four angles, then of five angles, and so on; we shall prove by induction that the formulae for the sine and the cosine of the sum of n angles $A_1, A_2 \dots A_n$ are

$$\sin(A_1+A_2+\dots+A_n)=S_1-S_3+S_5-\dots\dots\dots(28),$$

$$\cos(A_1+A_2+\dots+A_n)=S_0-S_2+S_4-\dots\dots\dots(29),$$

where S_r denotes the sum of the products of the sines of r of the angles and the cosines of the remaining $n-r$ angles, the r angles being chosen from the n angles in every possible way, thus

$$S_0=\cos A_1\cos A_2\dots\cos A_n$$

$$S_1=\sin A_1\cos A_2\dots\cos A_n+\cos A_1\sin A_2\cos A_3\dots\cos A_n+\dots.$$

The formulae (28), (29) agree with the formulae (1), (2), and (24), (25), for the cases $n=2$, $n=3$; assuming the formulae to hold for n angles, we shall shew that they hold for $n+1$ angles; we have

$$\begin{aligned}\sin(A_1+A_2+\dots+A_n+A_{n+1})\\ &=\sin(A_1+\dots+A_n)\cos A_{n+1}+\cos(A_1+\dots+A_n)\sin A_{n+1}\\ &=\cos A_{n+1}(S_1-S_3+S_5\dots)+\sin A_{n+1}(S_0-S_2+S_4\dots),\end{aligned}$$

now let S_r' denote the sum of the products of the sines of r of the angles $A_1, A_2 \dots A_{n+1}$, and of the cosines of the remaining $n+1-r$ angles, the r angles being chosen from the $n+1$ in every possible way, then we have

$$S_1'=S_1\cos A_{n+1}+S_0\sin A_{n+1},$$

for in $S_1\cos A_{n+1}$ there is in each term the sine of one of the angles $A_1, A_2 \dots A_n$, and in each term of $S_0\sin A_{n+1}$ there is only $\sin A_{n+1}$.

Similarly

$$S_3' = S_3 \cos A_{n+1} + S_2 \sin A_{n+1}$$
$$S_5' = S_5 \cos A_{n+1} + S_4 \sin A_{n+1}$$

....

....

hence $\sin(A_1 + \ldots + A_{n+1}) = S_1' - S_3' + S_5' \ldots.$

We may similarly shew that

$$\cos(A_1 + \ldots + A_{n+1}) = S_0' - S_2' + S_4' \ldots,$$

thus if the formulae (28), (29) hold for n angles, they also hold for $n+1$; and they have been shewn to hold for $n = 2, 3$, hence they are true generally.

These formulae may be written in the form

$$\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (t_1 - t_3 + t_5 \ldots),$$
$$\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (1 - t_2 + t_4 \ldots),$$

where t_r denotes the sum of the products of $\tan A_1$, $\tan A_2 \ldots \tan A_n$, taken r together; hence by division we have

$$\tan(A_1 + A_2 + \ldots + A_n) = \frac{t_1 - t_3 + t_5 \ldots}{1 - t_2 + t_4 \ldots} \quad \ldots\ldots(30),$$

which is the formula for the tangent of the sum of n angles, in terms of the tangents of those angles.

The formula (30) may also be proved independently. Assuming it to hold for n angles, we shall prove that it holds for $n+1$; we have

$$\tan(A_1+A_2+\ldots+A_{n+1}) = \frac{\tan(A_1+A_2+\ldots+A_n)+\tan A_{n+1}}{1-\tan(A_1+A_2+\ldots+A_n)\tan A_{n+1}}$$
$$= \frac{(t_1-t_3+t_5-\ldots)+\tan A_{n+1}(1-t_2+t_4+\ldots)}{(1-t_2+t_4-\ldots)-\tan A_{n+1}(t_1-t_3+t_5-\ldots)}.$$

Now if t_r' denote the sum of the products of the tangents of r of the $n+1$ angles, we have then

$$t_1' = t_1 + \tan A_{n+1}$$
$$t_2' = t_2 + t_1 \tan A_{n+1}$$
$$t_3' = t_3 + t_2 \tan A_{n+1}$$

....

....

hence $\tan(A_1+A_2+\ldots+A_{n+1}) = \dfrac{t_1' - t_3' + t_5' - \ldots}{1 - t_2' + t_4' - \ldots};$

since the formula (30) holds for $n = 2, 3$, it therefore holds for $n=4$, and generally.

Expression for a product of sines or of cosines as the sum of sines or cosines.

50. We may obtain formulae which exhibit the product of the sines or of the cosines of any number of angles as the sum of sines or cosines of composite angles; we have

$$2\sin A_1 \sin A_2 = \cos(A_1 - A_2) - \cos(A_1 + A_2).$$

$$\begin{aligned}2^2 \sin A_1 \sin A_2 \sin A_3 &= 2\sin A_3 \cos(A_1 - A_2) - 2\sin A_3 \cos(A_1 + A_2)\\ &= \sin(A_1 - A_2 + A_3) + \sin(-A_1 + A_2 + A_3)\\ &\qquad + \sin(A_1 + A_2 - A_3) - \sin(A_1 + A_2 + A_3)\\ &= \Sigma \sin(-A_1 + A_2 + A_3) - \sin(A_1 + A_2 + A_3).\end{aligned}$$

$$\begin{aligned}&2^3 \sin A_1 \sin A_2 \sin A_3 \sin A_4\\ &= 2\sin(A_1 - A_2 + A_3)\sin A_4 + \ldots - 2\sin(A_1 + A_2 + A_3)\sin A_4\\ &= \cos(A_1 - A_2 + A_3 - A_4) - \cos(A_1 - A_2 + A_3 + A_4)\\ &\quad + \cos(-A_1 + A_2 + A_3 - A_4) - \cos(-A_1 + A_2 + A_3 + A_4)\\ &\quad + \cos(\ \ A_1 + A_2 - A_3 - A_4) - \cos(\ \ A_1 + A_2 - A_3 + A_4)\\ &\quad - \cos(\ \ A_1 + A_2 + A_3 - A_4) + \cos(\ \ A_1 + A_2 + A_3 + A_4)\\ &= \cos(A_1 + A_2 + A_3 + A_4) - \Sigma \cos(A_1 + A_2 + A_3 - A_4)\\ &\quad + \tfrac{1}{2}\Sigma \cos(A_1 + A_2 - A_3 - A_4).\end{aligned}$$

Similarly

$$2\cos A_1 \cos A_2 = \cos(A_1 - A_2) + \cos(A_1 + A_2).$$

$$\begin{aligned}&2^2 \cos A_1 \cos A_2 \cos A_3\\ &= 2\cos(A_1 - A_2)\cos A_3 + 2\cos(A_1 + A_2)\cos A_3\\ &= \cos(-A_1 + A_2 + A_3) + \cos(A_1 - A_2 + A_3)\\ &\qquad\qquad + \cos(A_1 + A_2 - A_3) + \cos(A_1 + A_2 + A_3)\\ &= \Sigma \cos(-A_1 + A_2 + A_3) + \cos(A_1 + A_2 + A_3).\end{aligned}$$

$$\begin{aligned}&2^3 \cos A_1 \cos A_2 \cos A_3 \cos A_4\\ &= \Sigma \cos(-A_1 + A_2 + A_3 + A_4) + \tfrac{1}{2}\Sigma \cos(A_1 + A_2 - A_3 - A_4)\\ &\qquad\qquad + \cos(A_1 + A_2 + A_3 + A_4).\end{aligned}$$

The general formulae for n angles are the following:

$$\begin{aligned}&(-1)^{\frac{n}{2}}\, 2^{n-1} \sin A_1 \sin A_2 \ldots \sin A_n\\ &\qquad = C_n - C_{n-1} + C_{n-2} - \ldots + (-1)^{\frac{n}{2}}\, \tfrac{1}{2} C_{\frac{1}{2}n} \ldots\ldots\ldots\ldots\ldots (31)\end{aligned}$$

when n is even,

where C_{n-r} is the sum of the cosines of the sum of $n-r$ of the angles taken positively and the remaining r taken negatively, the negative angles being taken in every combination; and when n is odd

$$(-1)^{\frac{n-1}{2}} 2^{n-1} \sin A_1 \sin A_2 \dots \sin A_n$$
$$= D_n - D_{n-1} + D_{n-2} - \dots + (-1)^{\frac{n-1}{2}} D_{\frac{1}{2}(n+1)} \dots\dots\dots (32),$$

where D_{n-r} denotes the sum of the sines of the sum of $n-r$ of the angles taken positively and the remaining r taken negatively;

$$2^{n-1} \cos A_1 \cos A_2 \dots \cos A_n$$
$$= C_n + C_{n-1} + C_{n-2} + \dots + \tfrac{1}{2} C_{\frac{1}{2}n} \dots\dots\dots\dots (33)$$

when n is even, and

$$2^{n-1} \cos A_1 \cos A_2 \dots \cos A_n$$
$$= C_n + C_{n-1} + \dots + C_{\frac{1}{2}(n+1)} \dots\dots\dots\dots\dots\dots (34)$$

when n is odd.

These formulae (31), (32), (33), (34) have been proved above, in the cases $n = 2$, 3, 4, and may now be proved generally by induction; assume the formula (31) to hold for n, multiply it by $2 \sin A_{n+1}$, and replace any term $2C_{n-r} \sin A_{n+1}$ by a sum of sines, we then obtain for the product

$$(-1)^{\frac{n}{2}} 2^n \sin A_1 \sin A_2 \dots \sin A_n \sin A_{n+1}$$

the expression

$$D'_{n+1} - D'_n + \dots + (-1)^{\frac{n}{2}} D'_{\frac{1}{2}(n+2)},$$

where D'_r denotes the sum of the sines of the sum of r of the $n+1$ angles taken positively and the remainder taken negatively; this is what (32) becomes when n is changed into $n+1$; proceed again in a similar manner with this result, we then shew that the product

$$(-1)^{\frac{n+2}{2}} 2^{n+1} \sin A_1 \dots \sin A_{n+2}$$

is equal to

$$C''_{n+2} - C''_{n+1} + \dots + (-1)^{\frac{n+2}{2}} \tfrac{1}{2} C''_{\frac{1}{2}(n+2)},$$

where C''_r refers to $n+2$ angles; thus the formula (31) is proved for the value $n+2$, if we assume (31) and (32), for the value n; similarly we may shew that (32) holds for $n+2$; therefore as these formulae have been proved for $n = 3$, 4, they hold generally. The formulae (33), (34), for the products of a number of cosines, may be proved in a similar manner.

Example. *Prove that for* n *angles* $\alpha, \beta, \gamma, \delta\ldots$

$$\Sigma \sin(\alpha \pm \beta \pm \gamma \pm \delta \pm \ldots) = 2^{n-1} \sin\alpha \cos\beta \cos\gamma \cos\delta\ldots,$$

$$\Sigma \cos(\alpha \pm \beta \pm \gamma \pm \delta \pm \ldots) = 2^{n-1} \cos\alpha \cos\beta \cos\gamma \cos\delta\ldots,$$

where Σ *implies summation extending to all possible arrangements of the signs indicated in the* n − 1 *ambiguities.*

Formulae for the circular functions of multiple angles.

51. If, in the addition formulae which we have obtained for two and more angles, we suppose each angle equal to A, we obtain the formulae

$$\sin 2A = 2\sin A\cos A \ldots\ldots\ldots\ldots(35),$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1\ldots(36),$$

$$\sin 3A = 3\sin A\cos^2 A - \sin^3 A,$$

or $$\sin 3A = 3\sin A - 4\sin^3 A \ldots\ldots\ldots\ldots(37),$$

$$\cos 3A = \cos^3 A - 3\cos A\sin^2 A,$$

or $$\cos 3A = 4\cos^3 A - 3\cos A\ldots\ldots\ldots\ldots(38),$$

$$\sin nA = n\sin A\cos^{n-1}A - \frac{n(n-1)(n-2)}{3!}\sin^3 A\cos^{n-3}A + \ldots(39),$$

$$\cos nA = \cos^n A - \frac{n(n-1)}{2!}\sin^2 A\cos^{n-2}A + \frac{n(n-1)(n-2)(n-3)}{4!}\sin^4 A\cos^{n-4}A - \ldots(40).$$

These last formulae (39), (40) follow from (28), (29), since S_r in Art. 49 contains as many terms as there are combinations of n things taken r together, and becomes equal to

$$\frac{n(n-1)\ldots(n-r+1)}{r!}\sin^r A\cos^{n-r}A.$$

The formulae (39), (40) may also be written

$$\sin nA = \cos^n A\left\{n\tan A - \frac{n(n-1)(n-2)}{3!}\tan^3 A + \ldots\right\},$$

$$\cos nA = \cos^n A\left\{1 - \frac{n(n-1)}{2!}\tan^2 A + \frac{n(n-1)(n-2)(n-3)}{4!}\tan^4 A - \ldots\right\}.$$

We find also, from (9), (26), and (30),

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \quad \text{..........} (41),$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A} \quad \text{..........} (42),$$

$$\tan nA = \frac{n\tan A - \frac{n(n-1)(n-2)}{3!}\tan^3 A + \dots}{1-\frac{n(n-1)}{2!}\tan^2 A + \dots} \quad \text{......} (43).$$

We have thus obtained formulae for the circular functions of the multiples of an angle in terms of those of the angle itself.

It should be noticed that each of the sequences of numbers

$$\sin A,\ \sin 2A,\ \sin 3A \dots\dots\dots$$
$$\cos A,\ \cos 2A,\ \cos 3A \dots\dots\dots$$

is a recurring one; for we have

$$\sin(n+1)A = 2\cos A \,.\, \sin nA - \sin(n-1)A,$$
$$\cos(n+1)A = 2\cos A \,.\, \cos nA - \cos(n-1)A;$$

thus each term of either sequence is obtained by multiplying the preceding one by $2\cos A$, and then subtracting the term next but one preceding. By this means the terms of the sequences may be successively calculated, if we assume the formulae (35) and (36).

The scale of relation of either of the series

$$1 + x\sin A + x^2 \sin 2A + \dots\dots, \qquad 1 + x\cos A + x^2\cos 2A + \dots\dots$$

is consequently $1 - 2x\cos A + x^2$.

Expressions for the powers of a sine or cosine as sines or cosines of multiple angles.

52. In order to obtain expressions for a power of the cosine or sine of an angle, in terms of cosines or sines of multiples of that angle, we must make all the angles equal to one another in the formulae of Art. 50; we thus obtain the formulae

$$2\sin^2 A = 1 - \cos 2A,$$
$$4\sin^3 A = 3\sin A - \sin 3A,$$
$$8\sin^4 A = \cos 4A - 4\cos 2A + 3,$$
$$2\cos^2 A = 1 + \cos 2A,$$
$$4\cos^3 A = 3\cos A + \cos 3A,$$
$$8\cos^4 A = \cos 4A + 4\cos 2A + 3,$$

$$(-1)^{\frac{n}{2}}\,2^{n-1}\sin^n A = \cos nA - n\cos(n-2)A + \frac{n(n-1)}{2\,!}\cos(n-4)A - \ldots$$
$$+(-1)^{\frac{n}{2}}\tfrac{1}{2}\frac{n\,!}{\tfrac{1}{2}n\,!\,\tfrac{1}{2}n\,!}\,\ldots\ldots\ldots(44)$$
$$(n \text{ even}),$$

$$(-1)^{\frac{n-1}{2}}\,2^{n-1}\sin^n A = \sin nA - n\sin(n-2)A + \frac{n(n-1)}{2\,!}\sin(n-4)A - \ldots$$
$$+(-1)^{\frac{n-1}{2}}\frac{n\,!}{\tfrac{1}{2}(n-1)\,!\,\tfrac{1}{2}(n+1)\,!}\sin A\ldots\ldots(45)$$
$$(n \text{ odd}),$$

$$2^{n-1}\cos^n A = \cos nA + n\cos(n-2)A + \frac{n(n-1)}{2\,!}\cos(n-4)A + \ldots$$
$$+\tfrac{1}{2}\frac{n\,!}{\tfrac{1}{2}n\,!\,\tfrac{1}{2}n\,!}\,\ldots\ldots\ldots\ldots\ldots(46)$$
$$(n \text{ even}),$$

$$2^{n-1}\cos^n A = \cos nA + n\cos(n-2)A + \frac{n(n-1)}{2\,!}\cos(n-4)A + \ldots$$
$$+\frac{n\,!}{\tfrac{1}{2}(n-1)\,!\,\tfrac{1}{2}(n+1)\,!}\cos A\ldots\ldots(47)$$
$$(n \text{ odd}).$$

The formulae (44), (45) may be deduced from (46), (47) by writing $90° - A$ for A, or conversely.

Relations between inverse functions.

53. Corresponding to the addition formulae of this Chapter, formulae involving the inverse circular functions may be found. Thus in formulae (1) and (3), put $\cos A = a$, $\cos B = b$, then we have

$$\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}\{ab \mp \sqrt{1-a^2}\sqrt{1-b^2}\};$$

similarly from (2) and (4), we have

$$\sin^{-1} a \pm \sin^{-1} b = \sin^{-1}\{a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\}.$$

From (9), (10), (11), and (12) we obtain

$$\tan^{-1} a \pm \tan^{-1} b = \tan^{-1}\frac{a \pm b}{1 \mp ab},$$

$$\cot^{-1} a \pm \cot^{-1} b = \cot^{-1}\frac{ab \mp 1}{b \pm a}.$$

Again from (26) and (30), we have

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \tan^{-1}\left(\frac{a+b+c-abc}{1-bc-ca-ab}\right),$$

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \dots + \tan^{-1} a_n = \tan^{-1}\left(\frac{s_1 - s_3 + s_5 \dots}{1 - s_2 + s_4 \dots}\right),$$

where s_r is the sum of the products of $a_1 a_2 \dots a_n$ taken r together.

It should be observed that, in these formulae, the particular values to be assigned to all except one of the inverse functions are arbitrary, but the particular value of that one is determined when the values of the others have been assigned. Moreover if in a formula involving, for instance, three inverse functions, two of them have their principal values, it is not necessarily the case that the third has its principal value. For example, in the formula

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} (a + b)/(1 - ab),$$

if $\tan^{-1} a$, $\tan^{-1} b$ are both positive and have their principal values, that is, values between 0 and $\frac{1}{2}\pi$, and if their sum is greater than $\frac{1}{2}\pi$, this sum is not the principal value of

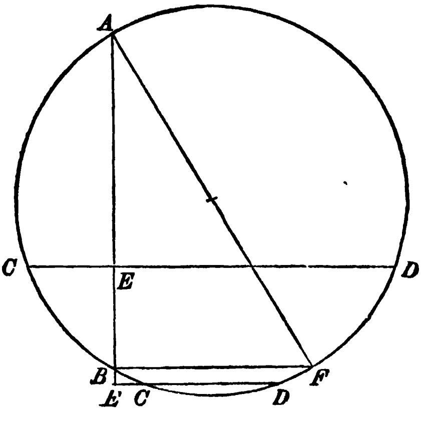
$$\tan^{-1} (a + b)/(1 - ab);$$

this principal value is an angle between 0 and $-\frac{1}{2}\pi$, which has the same tangent as the sum of $\tan^{-1} a$ and $\tan^{-1} b$.

Geometrical proofs of formulae.

54. Direct Geometrical proofs may be given of many of the formulae of this Chapter; we shall give three examples of such proofs. It should be remembered that such proofs often hold only for a limited range of the angles.

(1) To prove the formulae $\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.



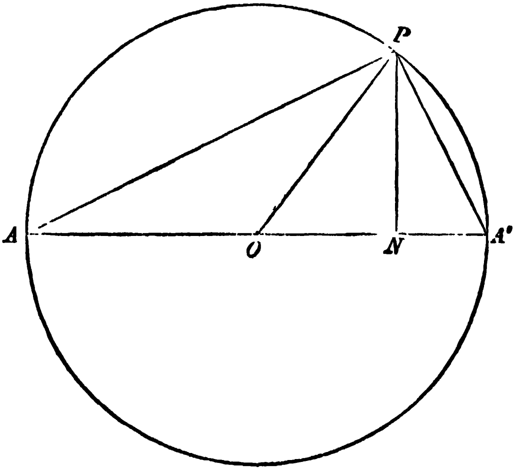
Let AB, CD be two chords of a circle at right angles, and let the angles ADE, BDE be denoted by A and B; since $AE.EB=CE.ED$, we have

$$\frac{\dfrac{AE\pm EB}{ED}}{1-\dfrac{AE}{ED}\cdot\dfrac{EB}{ED}}=\frac{AE\pm EB}{ED\mp EC}=\frac{AB}{BF},$$

whence $$\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}=\tan(A\pm B).$$

(2) To prove the formulae

$$\sin 2A=2\sin A\cos A,\qquad \cos 2A=\cos^2 A-\sin^2 A.$$



Let AOA' be the diameter of a circle, and let $PAA'=A$, then $POA'=2A$; draw PN perpendicular to AA'.

Then $\sin 2A=\dfrac{PN}{OP}$, now $PN.AA'=2\Delta APA'=AP.PA'$,

therefore $$\sin 2A=\frac{AP.A'P}{OP.AA'}=\frac{AA'^2\sin A\cos A}{OP.AA'}=2\sin A\cos A,$$

also $$\cos 2A=\frac{ON}{OP}=\frac{AN^2-A'N^2}{2.AA'.OP}=\frac{AP^2-A'P^2}{AA'^2}=\cos^2 A-\sin^2 A.$$

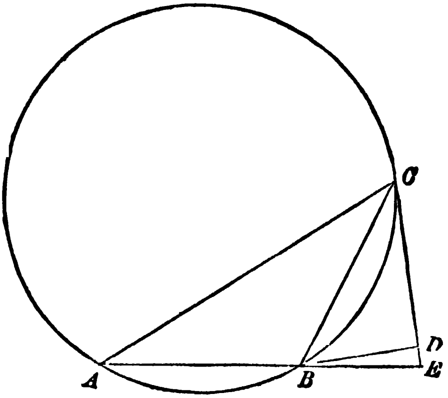
(3) To prove the formulae

$$\sin 3A=3\sin A-4\sin^3 A,\quad \cos 3A=4\cos^3 A-3\cos A.$$

Let $CAB=ACB=A$; let AB meet the tangent at C to the circle round the triangle ABC in E; draw BD perpendicular to CE.

The angle BED is $3A$, or $180° - 3A$. Now

$$\frac{AE}{BE} = \frac{\Delta ACE}{\Delta BCE} = \frac{AC^2}{BC^2} = 4\cos^2 A\,;$$



therefore $$\frac{AB}{BE} = 4\cos^2 A - 1 = 3 - 4\sin^2 A\,;$$

hence $$\sin 3A = \frac{BD}{BE} = \frac{BD}{AB}\cdot\frac{AB}{BE} = 3\sin A - 4\sin^3 A,$$

and $$\cos 3A = \mp\frac{DE}{BE} = \frac{DC}{BE} - \frac{EC}{BE} = \frac{DC}{BC}\cdot\frac{BC}{BE} - \frac{AC}{AB}$$

$$= \cos A\,(4\cos^2 A - 1) - 2\cos A = 4\cos^3 A - 3\cos A.$$

The proofs in (1) and (3) were given by Mr Hart in the *Messenger of Mathematics*, Vol. IV.

EXAMPLES.

Prove geometrically the formulae

(1) $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$.

(2) $\tan(45° + A) - \tan(45° - A) = 2\tan 2A$.

(3) $\sin A \sin B = \sin^2 \frac{1}{2}(A + B) - \sin^2 \frac{1}{2}(A - B)$.

(4) $\sin^2 \alpha + \sin^2 \beta = \sin^2(\alpha + \beta) - 2\sin\alpha\sin\beta\cos(\alpha + \beta)$.

(5) $\tan^{-1}\dfrac{m}{n} - \tan^{-1}\dfrac{m - n}{m + n} = \dfrac{\pi}{4}$.

(6) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$, *where* $A + B + C = 180°$

(7) $\sin A + \sin B - \sin C = 4\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C$, *where* $A + B + C = 180°$.

(8) $\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta$.

(9) $\cos 36° - \sin 18° = \frac{1}{2}$.

EXAMPLES ON CHAPTER IV.

Prove the identities in Examples 1—15:

1. $\cos^2 A + \cos^2(120° + A) + \cos^2(120° - A) = \frac{3}{2}$.

2. $(\cos A + \sin A)^4 + (\cos A - \sin A)^4 = 3 - \cos 4A$.

3. $\sin 3A \sin^3 A + \cos 3A \cos^3 A = \cos^3 2A$.

4. $4\cos^3 A \sin 3A + 4\sin^3 A \cos 3A = 3\sin 4A$.

5. $\sin^3 A + \sin^3(120° + A) - \sin^3(120° - A) = -\frac{3}{4}\sin 3A$.

6. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$.

7. $16\cos^5 A - \cos 5A = 5\cos A\,(1 + 2\cos 2A)$.

8. $\operatorname{cosec}(m+n)x \operatorname{cosec} mx \operatorname{cosec} nx - \cot(m+n)x \cot mx \cot nx$
$= \cot mx + \cot nx - \cot(m+n)x$

9. $\Sigma \cos A\,(\cos 3B - \cos 3C)$
$= 4(\cos B - \cos C)(\cos C - \cos A)(\cos A - \cos B)(\cos A + \cos B + \cos C)$.

10. $\Sigma \sin A\,(\sin^2 B + \sin^2 C)\sin(B - C)$
$= \sin(B - C)\sin(C - A)\sin(A - B)\sin(A + B + C)$.

11. $\tan(A + 60°)\tan(A - 60°) + \tan A \tan(A + 60°) + \tan(A - 60°)\tan A = -3$.

12. $\cot(A + 60°)\cot(A - 60°) + \cot A \cot(A + 60°) + \cot(A - 60°)\cot A = -3$.

13. $\dfrac{\cos 3A}{\cos A} - \dfrac{\cos 6A}{\cos 2A} + \dfrac{\cos 9A}{\cos 3A} - \dfrac{\cos 18A}{\cos 6A}$
$= 2\{\cos 2A - \cos 4A + \cos 6A - \cos 12A\}$.

14. $\Sigma \dfrac{\sin(B + C + D - A)}{\sin(A - B)\sin(A - C)\sin(A - D)} = 0$.

15. $\dfrac{\cos 4A}{\sin A \sin(A - B)\sin(A - C)} + \dfrac{\cos 4B}{\sin B \sin(B - C)\sin(B - A)}$
$+ \dfrac{\cos 4C}{\sin C \sin(C - A)\sin(C - B)} = 8\sin(A + B + C) + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

If $A + B + C = \pi$, prove the relations in Examples 16—27:

16. $\Sigma \tan A \cot B \cot C = \Sigma \tan A - 2\Sigma \cot A$.

17. $\Sigma \cot A = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

18. $\Sigma \sin(B - C)\cos^3 A = -\sin(B - C)\sin(C - A)\sin(A - B)$.

19. $\Sigma(\sin B + \sin C)(\cos C + \cos A)(\cos A + \cos B)$
$= (\sin B + \sin C)(\sin C + \sin A)(\sin A + \sin B)$.

20. $\Sigma \sin A \cos(A - B)\cos(A - C) = 3\sin A \sin B \sin C + \sin 2A \sin 2B \sin 2C$.

21. $\Sigma \sin 2B \sin 2C = 4\{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C$
$+ \cos A \cos B \cos C\}$.

22. $\Sigma \cos 2A (\tan B - \tan C)$

$$= -2 \sin (B-C) \sin (C-A) \sin (A-B) \sec A \sec B \sec C.$$

23. $\Sigma \cos^2 A (\sin 2B + \sin 2C) = 2 \sin A \sin B \sin C.$

24. $\Sigma \cos A \sin 3A = \{\Sigma \sin 2A\} \{\frac{3}{2} + \Sigma \cos 2A\}.$

25. $(\sin A + \sin B + \sin C)(-\sin A + \sin B + \sin C)(\sin A - \sin B + \sin C)$

$$(\sin A + \sin B - \sin C) = 4 \sin^2 A \sin^2 B \sin^2 C.$$

26. $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = 0.$

27. $\Sigma \operatorname{cosec} B \operatorname{cosec} C \sec (B-C)$

$$= \sec (B-C) \sec (C-A) \sec (A-B) (3 + 8 \cos A \cos B \cos C).$$

28. Prove that, if $\alpha + \beta + \gamma = \frac{1}{2}\pi,$

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1.$$

29. Prove that

$$\frac{1}{1 + 2\cos(\frac{1}{3}\pi + \theta)} + \frac{1}{1 + 2\cos(\frac{1}{3}\pi - \theta)} = \frac{1}{2\cos\theta - 1}.$$

30. Prove that

$$\sin^2 (\theta + \alpha) + \sin^2 (\theta + \beta) - 2 \cos (\alpha - \beta) \sin (\theta + \alpha) \sin (\theta + \beta)$$

is independent of θ.

31. If $\tan \beta = \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$, shew that $\tan (\alpha - \beta) = (1 - n) \tan \alpha.$

32. If $\tan \phi = \dfrac{\sin \alpha \sin \theta}{\cos \theta - \cos \alpha}$, prove that $\tan \theta = \dfrac{\sin \alpha \sin \phi}{\cos \phi \pm \cos \alpha}.$

33. If $\sqrt{2} \cos A = \cos B + \cos^3 B, \qquad \sqrt{2} \sin A = \sin B - \sin^3 B,$

prove that $\pm \sin (A - B) = \cos 2B = \frac{1}{3}.$

34. Prove that

$$\frac{\cos 3\theta + \cos 3\phi}{2\cos(\theta - \phi) - 1} = (\cos \theta + \cos \phi) \cos (\theta + \phi) - (\sin \theta + \sin \phi) \sin (\theta + \phi).$$

35. If θ and ϕ satisfy the equation

$$\sin \theta + \sin \phi = \sqrt{3} (\cos \phi - \cos \theta),$$

then will $\sin 3\theta + \sin 3\phi = 0.$

36. Prove that $\tan 70° = \tan 20° + 2 \tan 40° + 4 \tan 10°.$

37. If $\dfrac{\cos^4 \alpha}{\cos^2 \beta} + \dfrac{\sin^4 \alpha}{\sin^2 \beta} = 1,$ then $\dfrac{\cos^4 \beta}{\cos^2 \alpha} + \dfrac{\sin^4 \beta}{\sin^2 \alpha} = 1.$

38. If $\cos (A + B) \sin (C + D) = \cos (A - B) \sin (C - D),$

then $\cot A \cot B \cot C = \cot D.$

39. If $\alpha + \beta + \gamma = \frac{1}{2}\pi,$ then

$$(\cos \alpha + \sin \alpha)(\cos \beta + \sin \beta)(\cos \gamma + \sin \gamma) = 2 (\cos \alpha \cos \beta \cos \gamma + \sin \alpha \sin \beta \sin \gamma).$$

40. If $$A+B+C=\pi \text{ and } \cos A=\cos B\cos C,$$
then will $$\cot B\cot C=\tfrac{1}{2}.$$

41. If $$4\sin^2\alpha\sin^2\beta\sin^2\gamma+\sin^4\alpha+\sin^4\beta+\sin^4\gamma-2\sin^2\beta\sin^2\gamma-2\sin^2\gamma\sin^2\alpha-2\sin^2\alpha\sin^2\beta=0,$$
show that $\alpha\pm\beta\pm\gamma$ is a multiple of π.

42. If $$\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma}{\tan\beta},$$
prove that $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0.$$

43. If $$\sec\alpha=\sec\beta\sec\gamma+\tan\beta\tan\gamma,$$
prove that
$$\sec\beta=\sec\gamma\sec\alpha+\tan\gamma\tan\alpha \text{ and } \sec\gamma=\sec\alpha\sec\beta+\tan\alpha\tan\beta.$$

44. If $$\frac{\sin^2\theta\cos\phi-\cos^2\theta\sin\phi}{\cos\theta\tan\alpha}=\frac{\sin^2\phi\cos\theta-\cos^2\phi\sin\theta}{\cos\phi\tan\beta}=\cos(\theta+\phi),$$
then $$\frac{\sin^2\alpha\cos\beta-\cos^2\alpha\sin\beta}{\cos\alpha\tan\theta}=\frac{\sin^2\beta\cos\alpha-\cos^2\beta\sin\alpha}{\cos\beta\tan\phi}=\cos(\alpha+\beta).$$

45. If A, B, C be positive angles such that $A+B+C=60^\circ$, prove that
$$\sec A\sec B\sec C+2\Sigma\tan B\tan C=2.$$

46. If
$$\frac{\cos(\theta+\beta)\cos(\theta+\gamma)+1}{\cos(\beta+\gamma)}=\frac{\cos(\theta+\gamma)\cos(\theta+\alpha)+1}{\cos(\gamma+\alpha)}=\frac{\cos(\theta+\alpha)\cos(\theta+\beta)+1}{\cos(\alpha+\beta)},$$
prove that $$\operatorname{cosec}(\beta-\alpha)\operatorname{cosec}(\gamma-\alpha)+\operatorname{cosec}(\gamma-\beta)\operatorname{cosec}(\alpha-\beta)+\operatorname{cosec}(\alpha-\gamma)\operatorname{cosec}(\beta-\gamma)=1.$$

47. Having given
$$\sin^4\theta+\sin^4\phi=14\sin^2\theta\sin^2\phi \text{ and } \sin\theta+\sin\phi=\sin\tfrac{1}{4}\pi,$$
prove that $$2\sin\theta=\sin(\tfrac{1}{3}\pi\pm\tfrac{1}{4}\pi)/\sin\tfrac{1}{3}\pi \text{ or } \cos(\tfrac{1}{3}\pi\pm\tfrac{1}{4}\pi)/\cos\tfrac{1}{3}\pi.$$

48. If $$\cos(A+B+C)=\cos A\cos B\cos C,$$
then $$8\sin(B+C)\sin(C+A)\sin(A+B)+\sin 2A\sin 2B\sin 2C=0.$$

49. If $\tan\theta+\tan\phi+\tan\psi=-\tan\theta\tan\phi\tan\psi=\tan(\theta+\phi+\psi)$, then either two of the angles θ, ϕ, ψ must be equal to $m\pi+\frac{1}{4}\pi$, $n\pi-\frac{1}{4}\pi$, or else one of them and also the sum of the other two must be multiples of π.

50. If $$\frac{\sin(\beta-\gamma)}{\cos\alpha}\cos(\theta-2\alpha)+\frac{\sin(\gamma-\alpha)}{\cos\beta}\cos(\theta-2\beta)+\frac{\sin(\alpha-\beta)}{\cos\gamma}\cos(\theta-2\gamma)=\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta),$$
prove that $$\cos\theta=\cos\alpha\cos\beta\cos\gamma.$$

51. If α, β, γ, δ be any four angles and $2\sigma=\alpha+\beta+\gamma+\delta$, then
$$\cos\alpha\cos\beta\cos\gamma\cos\delta+\sin\alpha\sin\beta\sin\gamma\sin\delta$$
$$=\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)\cos(\sigma-\delta)+\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)\sin(\sigma-\delta).$$

52. Prove that

$$\tan^{-1} x = 2\tan^{-1}\{\operatorname{cosec}\tan^{-1} x - \tan\cot^{-1} x\}.$$

53. Prove that

$$2\tan^{-1} x + 2\tan^{-1} y = \sin^{-1}\left\{\frac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right\}.$$

54. Prove that

$$\tan^{-1}\{\tfrac{1}{2}(\cos 2\alpha\sec 2\beta + \cos 2\beta\sec 2\alpha)\} = \tan^{-1}\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\} + \tan^{-1} 1.$$

55. Prove that

$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi = 2(\tan^{-1} 1 + \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}).$$

56. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi,$

then $x^2 + y^2 + z^2 + 2xyz = 1.$

57. If $\tan^{-1} y = 5\tan^{-1} x$, find y as an algebraical function of x; hence shew that tan 18° is a root of the equation $5z^4 - 10z^2 + 1 = 0.$

58. If $2\sigma = \alpha + \beta + \gamma$, shew that

$$\tan^{-1}\left(\frac{2\cos\alpha\cos\beta\cos\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1}\right) - \tan^{-1}[\tan\sigma\tan(\sigma-\alpha)\tan(\sigma-\beta)\tan(\sigma-\gamma)] = \tan^{-1} 1.$$

59. Prove that

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}} = \pi.$$

60. Prove that the algebraical equivalent of the equation

$$\sin^{-1} x \pm \sin^{-1} y \pm \sin^{-1} z \pm \sin^{-1} u = n\pi,$$

where n is an integer, is

$$\{4(s-x)(s-y)(s-z)(s-u) - (xy+zu)(xz+yu)(xu+yz)\}$$
$$\{4s(s-x-y)(s-x-z)(s-x-u) - (zu-xy)(yu-xz)(yz-xu)\} = 0,$$

where $2s = x + y + z + u.$

Solve the equations in Examples 61—75.

61. $\sin\theta + 2\cos\theta = 1.$

62. $\sin 5\theta = 16\sin^5\theta.$

63. $\sin 7\theta - \sin\theta = \sin 3\theta.$

64. $\tan 2\theta = 8\cos^2\theta - \cot\theta.$

65. $\tan(45° + A) = 3\tan(45° - A).$

66. $2\sin(\theta - \phi) = \sin(\theta + \phi) = 1.$

67. $\sec 4\theta - \sec 2\theta = 2.$

68. $\sin m\theta + \sin n\theta + \sin(m+n)\theta = 0$

69. $\sin\frac{n+1}{2}\theta+\sin\frac{n-1}{2}\theta=\cos\theta.$

70. $\tan\theta+\sec 2\theta=1.$

71. $2(\sin^4\theta+\cos^4\theta)=1.$

72. $\tan\theta+\tan 3\theta+\tan 5\theta=0.$

73. $\cot^{-1}x-\cot^{-1}(x+2)=15^\circ.$

74. $\left.\begin{array}{l} a\sin^{-1}x+b\cos^{-1}y=a \\ a\cos^{-1}x-b\sin^{-1}y=\beta \end{array}\right\}.$

75. $\operatorname{cosec} 4a-\operatorname{cosec} 4\theta=\cot 4a-\cot 4\theta.$

76. Draw graphs of the functions (*a*) $\sin x+\sin 2x$, (*b*) $\cos 2x/\cos x$.

77. Find all the solutions of the equation

$$a(\sin\theta-\cos a)=b(\sin a-\cos\theta).$$

78. If m be any integer, and $A+B+C=\pi$, shew that

$$\sin 2mA+\sin 2mB+\sin 2mC=(-1)^{m+1}4\sin mA\sin mB\sin mC,$$

$$\cos 2mA+\cos 2mB+\cos 2mC=(-1)^m 4\cos mA\cos mB\cos mC-1.$$

79. Prove that $$x^4+8xz+4z^2=4x^2y,$$

where

$$x=\sin A+\sin B+\sin C,\quad y=\sin B\sin C+\sin C\sin A+\sin A\sin B,$$
$$z=\sin A\sin B\sin C.$$

80. Prove that, if

$$\frac{1-\tan B\tan C}{\cos^2 A}+\frac{1-\tan C\tan A}{\cos^2 B}=2\frac{1-\tan A\tan B}{\cos^2 C},$$

either $\tan A$, $\tan C$, $\tan B$ are in arithmetic progression, or $A+B+C$ is an integral multiple of π.

81. If $\cos A=\cos\theta\sin\phi$, $\cos B=\cos\phi\sin\psi$, $\cos C=\cos\psi\sin\theta$, and $A+B+C=\pi$, prove that $\tan\theta\tan\phi\tan\psi=1$.

82. Solve the equations

$$4(\cos 3\theta+\cos 4\theta)(\cos 3\theta+\cos\theta)=1,$$

$$4(\cos 3\theta+\cos 5\theta)(\cos 6\theta+\cos 7\theta)=-1.$$

CHAPTER V.

THE CIRCULAR FUNCTIONS OF SUBMULTIPLE ANGLES.

Dimidiary Formulae.

55. IF in the formula (36) of the last Chapter we write $\frac{1}{2}\alpha$ for A, we have

$$\cos\alpha = \cos^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\alpha = 2\cos^2\tfrac{1}{2}\alpha - 1 = 1 - 2\sin^2\tfrac{1}{2}\alpha,$$

whence we have

$$1 - \cos\alpha = 2\sin^2\tfrac{1}{2}\alpha, \quad 1 + \cos\alpha = 2\cos^2\tfrac{1}{2}\alpha;$$

taking the square roots we obtain the following formulae for $\cos\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\alpha$, in terms of $\cos\alpha$,

$$\sin\tfrac{1}{2}\alpha = \pm\sqrt{\tfrac{1}{2}(1-\cos\alpha)}, \quad \cos\tfrac{1}{2}\alpha = \pm\sqrt{\tfrac{1}{2}(1+\cos\alpha)};$$

dividing one of these expressions by the other, we have also

$$\tan\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

These three formulae contain an ambiguity of sign; now if α is given, the three functions $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $\tan\frac{1}{2}\alpha$ have each a unique value, and the true expressions for them can therefore contain no ambiguity. The reason of the ambiguity in the three expressions obtained above is that they give the values of $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $\tan\frac{1}{2}\alpha$, not when α is given, but when $\cos\alpha$ is given; now, as we have proved in Art. 33, all the angles $2n\pi \pm \alpha$, where n is an integer, have the same cosine as α, hence formulae which give $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $\tan\frac{1}{2}\alpha$, in terms of $\cos\alpha$, will give these functions for all the angles included in the formula $\frac{1}{2}(2n\pi \pm \alpha)$, and not merely the values of $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $\tan\frac{1}{2}\alpha$ themselves.

To find the values which $\sin \frac{1}{2}(2n\pi \pm \alpha)$ may have, we must consider the two cases of an even and of an odd value of n; if $n=2m$

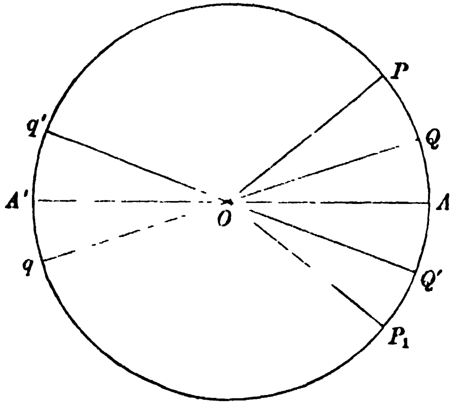
$$\sin \tfrac{1}{2}(4m\pi \pm \alpha) = \sin(\pm \tfrac{1}{2}\alpha) = \pm \sin \tfrac{1}{2}\alpha,$$

if $n = 2m + 1$

$$\sin \tfrac{1}{2}(4m\pi + 2\pi \pm \alpha) = \sin(\pi \pm \alpha) = \mp \sin \tfrac{1}{2}\alpha;$$

hence the values of $\sin \frac{1}{2}\alpha$ and $-\sin \frac{1}{2}\alpha$ are given by the formula which expresses $\sin \frac{1}{2}\alpha$ in terms of $\cos \alpha$. Similarly $\cos \frac{1}{2}(2n\pi \pm \alpha)$ and $\tan \frac{1}{2}(2n\pi \pm \alpha)$ can be shewn to have the values $\pm \cos \frac{1}{2}\alpha$, $\pm \tan \frac{1}{2}\alpha$, and thus the formulae which express $\cos \frac{1}{2}\alpha$, $\tan \frac{1}{2}\alpha$, in terms of $\cos \alpha$, will give the values of $\cos \frac{1}{2}\alpha$ and $-\cos \frac{1}{2}\alpha$, and of $\tan \frac{1}{2}\alpha$ and $-\tan \frac{1}{2}\alpha$, respectively. Thus the ambiguity of sign in the three formulae is accounted for.

56. The ambiguity of sign in the three formulae we have obtained may be illustrated geometrically.



If $AOP = \alpha$, and $AOP_1 = -\alpha$, the two sets of coterminal angles (OA, OP), (OA, OP_1) are the only ones which have the same cosine as α; if QOq, $Q'Oq'$ be the bisectors of the angles AOP, AOP_1, respectively, the bisector of any of the angles (OA, OP) is OQ or Oq, and of the angles (OA, OP_1) is OQ' or Oq'; hence the formulae for $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, $\tan \frac{1}{2}\alpha$, when $\cos \alpha$ is given, will give the sine, cosine, and tangent of all the four sets of coterminal angles (OA, OQ), (OA, Oq), (OA, OQ'), (OA, Oq'). The sines of the angles in the first and fourth sets are equal to $\sin \frac{1}{2}\alpha$, and in the second and third to $-\sin \frac{1}{2}\alpha$; the cosines of the angles in the first and third sets are equal to $\cos\frac{1}{2}\alpha$, and in the second and fourth to $-\cos\frac{1}{2}\alpha$; the tangents of the angles in the first and second sets are equal to $\tan\frac{1}{2}\alpha$, and in the third and fourth to $-\tan\frac{1}{2}\alpha$.

57. We shall now remove the ambiguities in the three formulae of Art. 55. The function $\sin\frac{1}{2}\alpha$ is positive or negative, according as $\frac{1}{2}\alpha$ lies between $2n\pi$ and $(2n+1)\pi$, or between $(2n+1)\pi$ and $(2n+2)\pi$, that is according as $\alpha/2\pi$ lies between $2n$ and $2n+1$, or between $2n+1$ and $2n+2$; hence we have the formula

$$\sin\tfrac{1}{2}\alpha = (-1)^p \sqrt{\tfrac{1}{2}(1-\cos\alpha)} \quad\text{.................}(1),$$

where p is the positive or negative integer algebraically next less than $\alpha/2\pi$.

The function $\cos\frac{1}{2}\alpha$ is positive or negative, according as $\frac{1}{2}\alpha$ lies between $2n\pi-\frac{1}{2}\pi$ and $2n\pi+\frac{1}{2}\pi$, or between $2n\pi+\frac{1}{2}\pi$ and $2n\pi+\frac{3}{2}\pi$, that is according as $\frac{1}{2}(\alpha+\pi)/\pi$ lies between $2n$ and $2n+1$, or between $2n+1$ and $2n+2$; hence

$$\cos\tfrac{1}{2}\alpha = (-1)^q \sqrt{\tfrac{1}{2}(1+\cos\alpha)} \quad\text{.................}(2),$$

where q is the integer algebraically next less than $\frac{1}{2}(\alpha+\pi)/\pi$.

We have also

$$\tan\tfrac{1}{2}\alpha = (-1)^{p-q}\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \quad\text{.................}(3);$$

the number $p-q$ is always either zero or ± 1.

58. If we write $\frac{1}{2}\alpha$ for A in the formula (35) of the last Chapter, we have

$$\sin\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha,$$

hence $$\tan\tfrac{1}{2}\alpha = \frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha} = \frac{\sin\alpha}{2\cos^2\frac{1}{2}\alpha} = \frac{2\sin^2\frac{1}{2}\alpha}{\sin\alpha}$$

Thus we have the two formulae

$$\tan\tfrac{1}{2}\alpha = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \quad\text{..............}(4),$$

which give $\tan\frac{1}{2}\alpha$ without ambiguity. These formulae give $\tan\frac{1}{2}\alpha$ when both $\sin\alpha$ and $\cos\alpha$ are given; now the formula $2n\pi+\alpha$ contains all the angles of which both the sine and cosine are the same as the sine and cosine of α, hence formulae for $\tan\frac{1}{2}\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$ give the tangents of all the angles $n\pi+\frac{1}{2}\alpha$, and all these angles have the same tangent $\tan \frac{1}{2}\alpha$; this accounts for the absence of ambiguity in the formulae (4).

59. We shall now obtain formulae for $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, and $\tan \frac{1}{2}\alpha$, in terms of $\sin\alpha$; we have

$$1 + \sin\alpha = 1 + 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha = (\sin\tfrac{1}{2}\alpha + \cos\tfrac{1}{2}\alpha)^2,$$

also
$$1 - \sin\alpha = 1 - 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha = (\sin\tfrac{1}{2}\alpha - \cos\tfrac{1}{2}\alpha)^2,$$

hence
$$\sin\tfrac{1}{2}\alpha + \cos\tfrac{1}{2}\alpha = \pm\sqrt{1+\sin\alpha},$$
$$\sin\tfrac{1}{2}\alpha - \cos\tfrac{1}{2}\alpha = \pm\sqrt{1-\sin\alpha};$$

therefore
$$\sin\tfrac{1}{2}\alpha = \tfrac{1}{2}\{\pm\sqrt{1+\sin\alpha} \pm \sqrt{1-\sin\alpha}\},$$
$$\cos\tfrac{1}{2}\alpha = \tfrac{1}{2}\{\pm\sqrt{1+\sin\alpha} \mp \sqrt{1-\sin\alpha}\}.$$

In each of the ambiguities either sign may be taken; we have, therefore, four values of $\sin\frac{1}{2}\alpha$, and four values of $\cos\frac{1}{2}\alpha$, in terms of $\sin\alpha$. Formulae which express $\sin\frac{1}{2}\alpha$ and $\cos\frac{1}{2}\alpha$ in terms of $\sin\alpha$ will give the sine and cosine respectively of all the angles included in the formula $\frac{1}{2}(n\pi + (-1)^n\alpha)$, for as we have shewn in Art. 33, the sines of all the angles $n\pi + (-1)^n\alpha$ have the value $\sin\alpha$. To find the sine and cosine of the angles $\frac{1}{2}(n\pi + (-1)^n\alpha)$ we must consider four cases.

(1) If $n = 4m$,

$$\tfrac{1}{2}(n\pi + (-1)^n\alpha) = 2m\pi + \tfrac{1}{2}\alpha;$$

the sine and cosine of these angles are $\sin\frac{1}{2}\alpha$ and $\cos\frac{1}{2}\alpha$ respectively.

(2) If $n = 4m + 1$,

$$\tfrac{1}{2}(n\pi + (-1)^n\alpha) = 2m\pi + \tfrac{1}{2}\pi - \tfrac{1}{2}\alpha;$$

the sine and cosine of these angles are $\cos\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\alpha$ respectively.

(3) If $n = 4m + 2$,

$$\tfrac{1}{2}(n\pi + (-1)^n\alpha) = 2m\pi + \pi + \tfrac{1}{2}\alpha;$$

the sine and cosine of these angles are $-\sin\frac{1}{2}\alpha$ and $-\cos\frac{1}{2}\alpha$ respectively.

(4) If $n = 4m + 3$,

$$\tfrac{1}{2}(n\pi + (-1)^n\alpha) = (2m+1)\pi + \tfrac{1}{2}\pi - \tfrac{1}{2}\alpha;$$

the sine and cosine of these angles are $-\cos\frac{1}{2}\alpha$ and $-\sin\frac{1}{2}\alpha$ respectively.

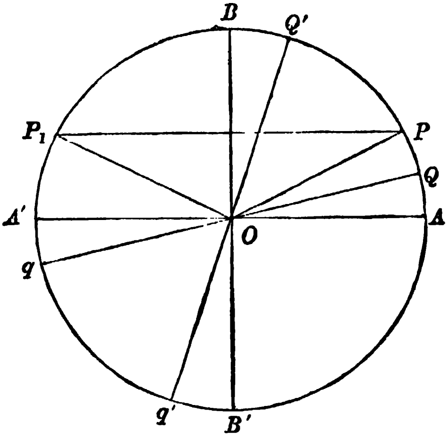
Thus we obtain four values $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $-\sin\frac{1}{2}\alpha$, $-\cos\frac{1}{2}\alpha$, by the formula which gives $\sin\frac{1}{2}\alpha$, and four values $\cos\frac{1}{2}\alpha$, $\sin\frac{1}{2}\alpha$, $-\cos\frac{1}{2}\alpha$, $-\sin\frac{1}{2}\alpha$, by the formula which gives $\cos\frac{1}{2}\alpha$.

The four sets of values of x and y which satisfy the equations

$$\left.\begin{array}{l}(x+y)^2 = 1+\sin\alpha\\(x-y)^2 = 1-\sin\alpha\end{array}\right\}$$

are $\left.\begin{array}{l}x=\sin\frac{1}{2}\alpha\\y=\cos\frac{1}{2}\alpha\end{array}\right\}$, $\left.\begin{array}{l}x=\cos\frac{1}{2}\alpha\\y=\sin\frac{1}{2}\alpha\end{array}\right\}$, $\left.\begin{array}{l}x=-\sin\frac{1}{2}\alpha\\y=-\cos\frac{1}{2}\alpha\end{array}\right\}$, $\left.\begin{array}{l}x=-\cos\frac{1}{2}\alpha\\y=-\sin\frac{1}{2}\alpha\end{array}\right\}$.

60. As in the preceding case, the ambiguities in the formulae of the last Article may be illustrated geometrically. Let $POA=\alpha$, $P_1OA=\pi-\alpha$, then the angles which have the same sine as α are the two sets of coterminal angles (OA, OP), (OA, OP_1); hence if QOq, $Q'Oq'$ be the bisectors of the angles AOP, AOP_1, the four sets of coterminal angles (OA, OQ), (OA, Oq), (OA, OQ'), (OA, Oq') will be the angles whose sine and cosine will be given by the formulae which express $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, when $\sin\alpha$ is given. We see that $Q'OB=\frac{1}{2}\alpha$, and $Q'OA=\frac{1}{2}(\pi-\alpha)$, hence the sines of these four sets of coterminal angles are $\sin\frac{1}{2}\alpha$, $-\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$, $-\cos\frac{1}{2}\alpha$, and their cosines are $\cos\frac{1}{2}\alpha$, $-\cos\frac{1}{2}\alpha$, $\sin\frac{1}{2}\alpha$, $-\sin\frac{1}{2}\alpha$; these are the four values of $\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\alpha$ respectively which are given by the two formulae.



61. We have

$$\sin\tfrac{1}{2}\alpha+\cos\tfrac{1}{2}\alpha=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\tfrac{1}{2}\alpha+\frac{1}{\sqrt{2}}\cos\tfrac{1}{2}\alpha\right)$$
$$=\sqrt{2}\sin\left(\tfrac{1}{2}\alpha+\tfrac{1}{4}\pi\right),$$

and similarly

$$\sin \tfrac{1}{2}\alpha - \cos \tfrac{1}{2}\alpha = \sqrt{2} \sin (\tfrac{1}{2}\alpha - \tfrac{1}{4}\pi);$$

hence $\sin \frac{1}{2}\alpha + \cos \frac{1}{2}\alpha$ is positive or negative, according as $\frac{\alpha}{2\pi} + \frac{1}{4}$ lies between $2n$ and $2n+1$, or between $2n+1$ and $2n+2$, and $\sin \frac{1}{2}\alpha - \cos \frac{1}{2}\alpha$ is positive or negative, according as $\frac{\alpha}{2\pi} - \frac{1}{4}$ lies between $2n$ and $2n+1$, or between $2n+1$ and $2n+2$; therefore

$$\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha = (-1)^p \sqrt{1 + \sin \alpha},$$

$$\sin \tfrac{1}{2}\alpha - \cos \tfrac{1}{2}\alpha = (-1)^q \sqrt{1 - \sin \alpha},$$

where p is the positive or negative integer algebraically next less than $\frac{\alpha}{2\pi} + \frac{1}{4}$, and q is the integer algebraically next less than $\frac{\alpha}{2\pi} - \frac{1}{4}$; we have then the three formulae

$$\sin \tfrac{1}{2}\alpha = \tfrac{1}{2}\{(-1)^p \sqrt{1+\sin\alpha} + (-1)^q \sqrt{1-\sin\alpha}\} \ldots\ldots (5),$$

$$\cos \tfrac{1}{2}\alpha = \tfrac{1}{2}\{(-1)^p \sqrt{1+\sin\alpha} - (-1)^q \sqrt{1-\sin\alpha}\} \ldots\ldots (6),$$

$$\tan \tfrac{1}{2}\alpha = \frac{(-1)^p \sqrt{1+\sin\alpha} + (-1)^q \sqrt{1-\sin\alpha}}{(-1)^p \sqrt{1+\sin\alpha} - (-1)^q \sqrt{1-\sin\alpha}} \ldots\ldots\ldots (7).$$

62. To express $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, $\tan \frac{1}{2}\alpha$ in terms of $\tan \alpha$, we have

$$\sin^2 \tfrac{1}{2}\alpha = \tfrac{1}{2}(1 - \cos\alpha)$$

$$= \tfrac{1}{2}\left(1 - \frac{1}{\pm\sqrt{1+\tan^2\alpha}}\right),$$

$$\cos^2 \tfrac{1}{2}\alpha = \tfrac{1}{2}\left(1 + \frac{1}{\pm\sqrt{1+\tan^2\alpha}}\right);$$

hence

$$\sin \tfrac{1}{2}\alpha = \pm\sqrt{\tfrac{1}{2}\left(1 - \frac{1}{\pm\sqrt{1+\tan^2\alpha}}\right)},$$

$$\cos \tfrac{1}{2}\alpha = \pm\sqrt{\tfrac{1}{2}\left(1 + \frac{1}{\pm\sqrt{1+\tan^2\alpha}}\right)},$$

and consequently

$$\tan \tfrac{1}{2}\alpha = \frac{\pm\sqrt{1+\tan^2\alpha} - 1}{\tan\alpha};$$

each of these formulae contains ambiguities. We leave to the student the discussion of these ambiguities, which should be made as in the previous cases.

It should be noticed that the values of $\tan \frac{1}{2}\alpha$ are the roots of the quadratic equation in $\tan \frac{1}{2}\alpha$,

$$\tan \alpha = \frac{2 \tan \frac{1}{2}\alpha}{1 - \tan^2 \frac{1}{2}\alpha},$$

obtained by replacing A by $\frac{1}{2}\alpha$, in the formula (41) of the last Chapter.

63. The functions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ can be expressed without ambiguity in terms of $\tan \frac{1}{2}\alpha$; for all the angles which have the same tangent as $\frac{1}{2}\alpha$ are included in the formula $n\pi + \frac{1}{2}\alpha$, and $2(n\pi + \frac{1}{2}\alpha)$ or $2n\pi + \alpha$ are angles which have all their circular functions the same as those of α. To find the expressions, we have

$$\sin \alpha = \frac{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha} = \frac{2 \tan \frac{1}{2}\alpha}{1 + \tan^2 \frac{1}{2}\alpha},$$

$$\cos \alpha = \frac{\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha}{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha} = \frac{1 - \tan^2 \frac{1}{2}\alpha}{1 + \tan^2 \frac{1}{2}\alpha},$$

hence also $$\tan \alpha = \frac{2 \tan \frac{1}{2}\alpha}{1 - \tan^2 \frac{1}{2}\alpha}.$$

EXAMPLES.

(1) *If* $2\cos\theta = \sqrt{1 - \sin 2\theta} - \sqrt{1 + \sin 2\theta}$, *shew that* θ *must lie between*

$$(8n+5)\frac{\pi}{4} \quad \textit{and} \quad (8n+7)\frac{\pi}{4},$$

where n *is an integer.*

(2) *Prove that* $$\frac{\cos \frac{1}{2}A}{\sqrt{1 + \sin A}} + \frac{\sin \frac{1}{2}A}{\sqrt{1 - \sin A}} = \sec A,$$

the radicals denoting positive numbers, provided A *lies between*

$$(4n - \tfrac{1}{2})\pi \quad \textit{and} \quad (4n + \tfrac{1}{2})\pi,$$

where n *is an integer. What are the signs in other cases?*

(3) *Prove that the four values of* $\dfrac{\sqrt{1 - \sin x} + 1}{\sqrt{1 + \sin x} - 1}$ *are*

$$\cot \tfrac{1}{4}x, \quad \tan \tfrac{1}{4}(\pi + x), \quad -\tan \tfrac{1}{4}x, \quad -\cot \tfrac{1}{4}(\pi + x).$$

(4) *If* $\sin 4A = a$, *shew that the four values of* $\tan A$ *are given by*

$$\frac{1}{a}\{(1+a)^{\frac{1}{2}} - 1\}\{1 + (1-a)^{\frac{1}{2}}\}.$$

(5) *In the formula* $\tan \frac{1}{2}A = \dfrac{\pm\sqrt{1 + \tan^2 A} - 1}{\tan A}$, *prove that the ambiguity of sign may be replaced by* $(-1)^m$, *where* m *is the greatest integer in* $(A + 90^\circ)/180^\circ$.

The circular functions of one-third of a given angle.

64. If we replace A, in the formulae (37), (38), (42) of the last Chapter, by $\frac{1}{3}\alpha$, we obtain the three equations

$$\sin\alpha = 3\sin\tfrac{1}{3}\alpha - 4\sin^3\tfrac{1}{3}\alpha \quad\text{..................(8)},$$

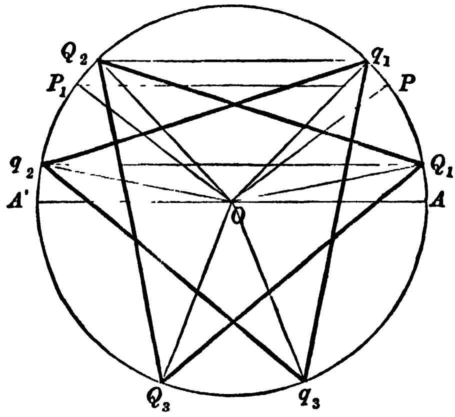
$$\cos\alpha = 4\cos^3\tfrac{1}{3}\alpha - 3\cos\tfrac{1}{3}\alpha \quad\text{..................(9)},$$

$$\tan\alpha = \frac{3\tan\frac{1}{3}\alpha - \tan^3\frac{1}{3}\alpha}{1 - 3\tan^2\frac{1}{3}\alpha} \quad\text{..................(10)};$$

we have thus, in each case, a cubic equation for determining a circular function of $\frac{1}{3}\alpha$, in terms of one of α. Hence if $\sin\alpha$ be given, we obtain three distinct values of $\sin\frac{1}{3}\alpha$; if $\cos\alpha$ be given, we obtain three distinct values of $\cos\frac{1}{3}\alpha$, and if $\tan\alpha$ be given, we obtain three distinct values of $\tan\frac{1}{3}\alpha$.

(1) In the case of the formula (8), we have $\sin\alpha$ given, and thus we shall obtain for $\sin\frac{1}{3}\alpha$ the values of the sines of one-third of all the angles (OA, OP), (OA, OP_1), which have the same sine as α. Let the trisectors of the angles (OA, OP) be OQ_1, OQ_2, OQ_3, so that $Q_1OA = \frac{1}{3}\alpha$, and $Q_1Q_2Q_3$ is an equilateral triangle, and

$$Q_2OA = \tfrac{2}{3}\pi + \tfrac{1}{3}\alpha, \quad Q_3OA = \tfrac{4}{3}\pi + \tfrac{1}{3}\alpha;$$



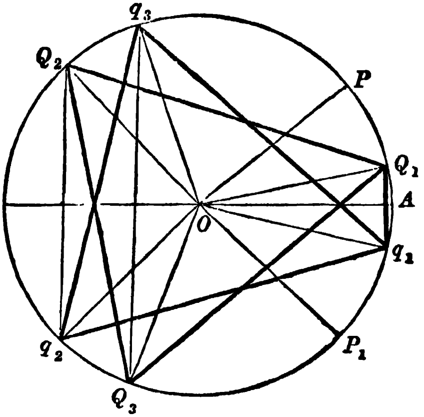
the trisectors of the angles (OA, OP_1) are Oq_1, Oq_2, Oq_3, where $q_1q_2q_3$ is an equilateral triangle, and $q_1OA = \frac{1}{3}(\pi - \alpha)$, so that

$$q_2OA = \pi - \tfrac{1}{3}\alpha, \quad q_3OA = \tfrac{5}{3}\pi - \tfrac{1}{3}\alpha.$$

We see at once that Q_2q_1, Q_1q_2, Q_3q_3 are parallel to OA; the sines of the two sets of coterminal angles (OA, OQ_1), (OA, Oq_2)

are $\sin\frac{1}{3}\alpha$, those of the sets (OA, OQ_2), (OA, Oq_1) are $\sin(\frac{2}{3}\pi+\frac{1}{3}\alpha)$, and those of (OA, OQ_3), (OA, Oq_3) are $\sin(\frac{4}{3}\pi+\frac{1}{3}\alpha)$; therefore the three roots of the cubic (8), in $\sin\frac{1}{3}\alpha$, will be $\sin\frac{1}{3}\alpha$, $\sin(\frac{1}{3}\pi-\frac{1}{3}\alpha)$, and $-\sin(\frac{1}{3}\pi+\frac{1}{3}\alpha)$.

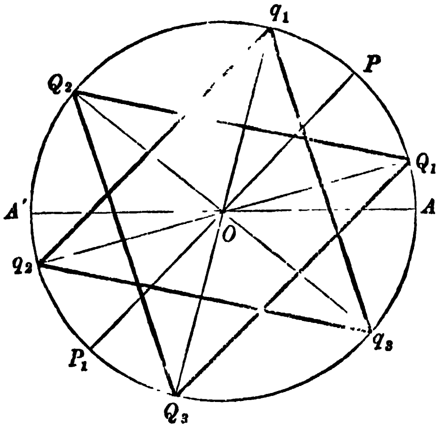
(2) In the case of the formula (9), the angles which have the same cosine as α are (OA, OP) and (OA, OP_1); let the trisectors of the first set of angles be the three lines OQ_1, OQ_2, OQ_3, where $Q_1OA=\frac{1}{3}\alpha$, and $Q_1Q_2Q_3$ is an equilateral triangle; the trisectors of the second set of angles are Oq_1, Oq_2, Oq_3, where $q_1OA=-\frac{1}{3}\alpha$, and $q_1q_2q_3$ is an equilateral triangle; we see at once that Q_1q_1, Q_2q_3, and Q_3q_2 are perpendicular to OA. The cosines of the two sets of angles (OA, OQ_1), (OA, Oq_1) are $\cos\frac{1}{3}\alpha$, those of the two sets (OA, OQ_2), (OA, Oq_2) are $\cos(\frac{2}{3}\pi+\frac{1}{3}\alpha)$, and those of the two sets (OA, OQ_3), (OA, Oq_3) are $\cos(\frac{4}{3}\pi+\frac{1}{3}\alpha)$; therefore the three roots of the cubic (9), in $\cos\frac{1}{3}\alpha$, are $\cos\frac{1}{3}\alpha$, $-\cos(\frac{1}{3}\pi-\frac{1}{3}\alpha)$ and $-\cos(\frac{1}{3}\pi+\frac{1}{3}\alpha)$.



(3) In the case of the formula (10), the angles which have the same tangent as α are (OA, OP) and (OA, OP_1). As before OQ_1, OQ_2, OQ_3, in the figure on page 72, are the trisectors of the first set of angles; the trisectors of the second set are Oq_1, Oq_2, Oq_3, where $q_1q_2q_3$ is an equilateral triangle, and $q_1OA=\frac{1}{3}(\pi+\alpha)$; we see that Q_1Oq_2, Q_2Oq_3, Q_3Oq_1 are diameters of the circle. The tangents of the sets (OA, OQ_1), (OA, Oq_2) are $\tan\frac{1}{3}\alpha$, of (OA, OQ_2), (OA, Oq_3) are $\tan(\frac{2}{3}\pi+\frac{1}{3}\alpha)$, and of (OA, OQ_3), (OA, Oq_1) are $\tan(\frac{4}{3}\pi+\frac{1}{3}\alpha)$, hence $\tan\frac{1}{3}\alpha$, $-\tan(\frac{1}{3}\pi-\frac{1}{3}\alpha)$, $\tan(\frac{1}{3}\pi+\frac{1}{3}\alpha)$ are the roots of the cubic (10), in $\tan\frac{1}{3}\alpha$.

We may express the results of this article thus; the roots of the cubic in x,

$$3x - 4x^3 = \sin\alpha, \text{ are } \sin\tfrac{1}{3}\alpha, \quad \sin\tfrac{1}{3}(\pi-\alpha), \quad -\sin\tfrac{1}{3}(\pi+\alpha),$$



those of the cubic

$$4x^3 - 3x = \cos\alpha, \text{ are } \cos\tfrac{1}{3}\alpha, \quad -\cos\tfrac{1}{3}(\pi-\alpha), \quad -\cos\tfrac{1}{3}(\pi+\alpha),$$

and those of the cubic

$$\tan\alpha(1-3x^2) = 3x - x^3, \text{ are } \tan\tfrac{1}{3}\alpha, \quad -\tan\tfrac{1}{3}(\pi-\alpha), \quad \tan\tfrac{1}{3}(\pi+\alpha).$$

Determination of the circular functions of certain angles.

65. The formulae of this Chapter may be applied to the determination of the circular functions of angles which are submultiples of angles whose circular functions are known.

(1) We have $\sin\frac{1}{4}\pi = \cos\frac{1}{4}\pi = 1/\sqrt{2}$;

hence from the formulae (1) and (2), of Art. 57,

$$\sin\tfrac{1}{8}\pi = \tfrac{1}{2}\sqrt{2-\sqrt{2}}, \qquad \cos\tfrac{1}{8}\pi = \tfrac{1}{2}\sqrt{2+\sqrt{2}},$$

$$\sin\tfrac{1}{16}\pi = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}, \quad \cos\tfrac{1}{16}\pi = \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},$$

and proceeding in this way, we can calculate $\sin\frac{1}{2^n}\pi$ and $\cos\frac{1}{2^n}\pi$.

(2) We have $\sin\frac{1}{6}\pi = 1/2$, $\cos\frac{1}{6}\pi = \sqrt{3}/2$;

hence from formulae (5) and (6), we have

$$\sin\tfrac{1}{12}\pi = \tfrac{1}{4}(\sqrt{6}-\sqrt{2}), \quad \cos\tfrac{1}{12}\pi = \tfrac{1}{4}(\sqrt{6}+\sqrt{2}),$$

the values obtained for sin 15°, cos 15° in Art. 34; proceeding in this way we calculate the sines and cosines of all the angles $\frac{\pi}{2^n \cdot 3}$.

(3) We have $\sin\frac{1}{5}\pi = 2\sin\frac{1}{10}\pi\cos\frac{1}{10}\pi$

and $\sin\frac{2}{5}\pi = 2\sin\frac{1}{5}\pi\cos\frac{1}{5}\pi$,

therefore $\sin\frac{1}{5}\pi\sin\frac{2}{5}\pi = 4\sin\frac{1}{5}\pi\cos\frac{1}{5}\pi\sin\frac{1}{10}\pi\cos\frac{1}{10}\pi$;

hence since $\sin\frac{2}{5}\pi = \cos\frac{1}{10}\pi$,

we have $4\cos\frac{1}{5}\pi\sin\frac{1}{10}\pi = 1$,

or $\sin\frac{3}{10}\pi - \sin\frac{1}{10}\pi = \frac{1}{2}$,

that is $\cos\frac{1}{5}\pi - \sin\frac{1}{10}\pi = \frac{1}{2}$,

also $(\cos\frac{1}{5}\pi + \sin\frac{1}{10}\pi)^2 = \frac{1}{4} + 1 = \frac{5}{4}$;

therefore $\cos\frac{1}{5}\pi + \sin\frac{1}{10}\pi = \frac{1}{2}\sqrt{5}$,

or $\sin\frac{1}{10}\pi = \frac{1}{4}(\sqrt{5}-1)$, $\cos\frac{1}{5}\pi = \frac{1}{4}(\sqrt{5}+1)$,

and hence $\cos\frac{1}{10}\pi = \frac{1}{4}\sqrt{10+2\sqrt{5}}$, $\sin\frac{1}{5}\pi = \frac{1}{4}\sqrt{10-2\sqrt{5}}$;

these values agree with those given in Art. 34.

It should be noticed that, if α is any angle of which the sine and cosine are known, then the sines and cosines of all angles of the form $m\alpha/2^n$, where m and n are positive integers, can be found in a form which involves only the extraction of radicals; for we have shewn how to find the functions of all angles of the form $\alpha/2^n$, and when these are known, the formulae of the last Chapter enable us to find $\sin\frac{m\alpha}{2^n}$ and $\cos\frac{m\alpha}{2^n}$.

66. We are now in a position to calculate the circular functions of all angles differing by 3° or $\pi/60$, commencing at 3°, and going up to 90°

We have
$$\begin{aligned}\sin 3^\circ &= \sin(18^\circ - 15^\circ)\\ &= \sin 18^\circ\cos 15^\circ - \cos 18^\circ\sin 15^\circ\\ &= \tfrac{1}{16}(\sqrt{6}+\sqrt{2})(\sqrt{5}-1) - \tfrac{1}{8}(\sqrt{3}-1)\sqrt{5+\sqrt{5}}\end{aligned}$$

similarly $\cos 3^\circ = \frac{1}{8}(\sqrt{3}+1)\sqrt{5+\sqrt{5}} + \frac{1}{16}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)$.

We have also

$$6^\circ = 36^\circ - 30^\circ,\quad 9^\circ = 45^\circ - 36^\circ,\quad 12^\circ = 30^\circ - 18^\circ,$$
$$21^\circ = 36^\circ - 15^\circ,\quad 24^\circ = 45^\circ - 21^\circ,\quad 27^\circ = 30^\circ - 3^\circ,$$
$$33^\circ = 45^\circ - 12^\circ,\quad 39^\circ = 45^\circ - 6^\circ,\quad 42^\circ = 45^\circ - 3^\circ;$$

hence we can calculate the sines and cosines of all the angles 3°, 6°,... up to 45°. It is then unnecessary to proceed farther, since the sine or cosine of an angle greater than 45° is the cosine or sine of its complement, which is less than 45°. The results of the calculation are given in the following table:

	sine
$3^\circ = \frac{1}{60}\pi$	$\frac{1}{16}\{(\sqrt{6}+\sqrt{2})(\sqrt{5}-1)-2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}\}$
$6^\circ = \frac{1}{30}\pi$	$\frac{1}{8}(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1)$
$9^\circ = \frac{1}{20}\pi$	$\frac{1}{8}(\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}})$
$12^\circ = \frac{1}{15}\pi$	$\frac{1}{8}(\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3})$
$15^\circ = \frac{1}{12}\pi$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$
$18^\circ = \frac{1}{10}\pi$	$\frac{1}{4}(\sqrt{5}-1)$
$21^\circ = \frac{7}{60}\pi$	$\frac{1}{16}\{2(\sqrt{3}+1)\sqrt{5-\sqrt{5}}-(\sqrt{6}-\sqrt{2})(\sqrt{5}+1)\}$
$24^\circ = \frac{2}{15}\pi$	$\frac{1}{8}(\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}})$
$27^\circ = \frac{3}{20}\pi$	$\frac{1}{8}(2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2})$
$30^\circ = \frac{1}{6}\pi$	$\frac{1}{2}$
$33^\circ = \frac{11}{60}\pi$	$\frac{1}{16}\{(\sqrt{6}+\sqrt{2})(\sqrt{5}-1)+2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}\}$
$36^\circ = \frac{1}{5}\pi$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$
$39^\circ = \frac{13}{60}\pi$	$\frac{1}{16}\{(\sqrt{6}+\sqrt{2})(\sqrt{5}+1)-2(\sqrt{3}-1)\sqrt{5-\sqrt{5}}\}$
$42^\circ = \frac{7}{30}\pi$	$\frac{1}{8}(\sqrt{30+6\sqrt{5}}-\sqrt{5}+1)$
$45^\circ = \frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$
$48^\circ = \frac{4}{15}\pi$	$\frac{1}{8}(\sqrt{10+2\sqrt{5}}+\sqrt{15}-\sqrt{3})$
$51^\circ = \frac{17}{60}\pi$	$\frac{1}{16}\{2(\sqrt{3}+1)\sqrt{5-\sqrt{5}}+(\sqrt{6}-\sqrt{2})(\sqrt{5}+1)\}$
$54^\circ = \frac{3}{10}\pi$	$\frac{1}{4}(\sqrt{5}+1)$
$57^\circ = \frac{19}{60}\pi$	$\frac{1}{16}\{2(\sqrt{3}+1)\sqrt{5+\sqrt{5}}-(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)\}$
$60^\circ = \frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$
$63^\circ = \frac{7}{20}\pi$	$\frac{1}{8}(2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2})$
$66^\circ = \frac{11}{30}\pi$	$\frac{1}{8}(\sqrt{30-6\sqrt{5}}+\sqrt{5}+1)$
$69^\circ = \frac{23}{60}\pi$	$\frac{1}{16}\{(\sqrt{6}+\sqrt{2})(\sqrt{5}+1)+2(\sqrt{3}-1)\sqrt{5-\sqrt{5}}\}$
$72^\circ = \frac{2}{5}\pi$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$
$75^\circ = \frac{5}{12}\pi$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$
$78^\circ = \frac{13}{30}\pi$	$\frac{1}{8}(\sqrt{30+6\sqrt{5}}+\sqrt{5}-1)$
$81^\circ = \frac{19}{20}\pi$	$\frac{1}{8}(\sqrt{10}+\sqrt{2}+2\sqrt{5-\sqrt{5}})$
$84^\circ = \frac{7}{15}\pi$	$\frac{1}{8}(\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}})$
$87^\circ = \frac{29}{60}\pi$	$\frac{1}{16}\{2(\sqrt{3}+1)\sqrt{5+\sqrt{5}}+(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)\}$

In this table, the sines of the angles 3°, 6°,... up to 87° are given; the cosines will be found by taking the sines of the complementary angles. The values of the surds in the above expressions are given to 24 decimal places in the *Messenger of Math.* Vol. VI., by Mr P. Gray. In Hutton's tables the values of these surds are given to 10 places of decimals. A complete table giving the tangents, secants, and cosecants of these angles, with the denominators in a rationalized form, will be found in Gelin's *Trigonometry.*

EXAMPLES ON CHAPTER V.

Prove the relations in Examples 1—8, where $A+B+C=180°$;

1. $$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}C}=\frac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}.$$

2. $$\sin(A-B)\sin(A-C)+\sin(B-C)\sin(B-A)+\sin(C-A)\sin(C-B)$$
$$=2\cos\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A-B)-2\sin\tfrac{3}{2}A\sin\tfrac{3}{2}B\sin\tfrac{3}{2}C.$$

3. $$\cos^4\tfrac{1}{2}A+\cos^4\tfrac{1}{2}B+\cos^4\tfrac{1}{2}C+2\cos A\cos^2\tfrac{1}{2}B\cos^2\tfrac{1}{2}C$$
$$+2\cos B\cos^2\tfrac{1}{2}C\cos^2\tfrac{1}{2}A+2\cos C\cos^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}B=8\cos^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}B\cos^2\tfrac{1}{2}C.$$

4. $$\Sigma\sin^3 A=3\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C+\cos\tfrac{3}{2}A\cos\tfrac{3}{2}B\cos\tfrac{3}{2}C.$$

5. $$\Sigma\operatorname{cosec}A\,(1+\cot B\cot C)$$
$$=\operatorname{cosec}A\operatorname{cosec}B\operatorname{cosec}C\,\{4\cos\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A-B)-1\}.$$

6. $$\Sigma\operatorname{cosec}A\,(1-\cot B\cot C)$$
$$=\tfrac{1}{2}\sec\tfrac{1}{2}A\sec\tfrac{1}{2}B\sec\tfrac{1}{2}C+\operatorname{cosec}A\operatorname{cosec}B\operatorname{cosec}C.$$

7. $$\Sigma\sin 2A\sin(B-C)$$
$$=16\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B).$$

8. $$\frac{\cos\frac{1}{2}A-\sin\frac{1}{2}B+\sin\frac{1}{2}C}{\cos\frac{1}{2}B+\sin\frac{1}{2}C-\sin\frac{1}{2}A}=\frac{1+\tan\frac{1}{4}A}{1+\tan\frac{1}{4}B}$$

9. Prove the identity
$$\frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}+\frac{\sin\frac{1}{2}(C-A)}{\sin\frac{1}{2}(C+A)}+\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$$
$$+\frac{\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(B+C)\sin\frac{1}{2}(C+A)\sin\frac{1}{2}(A+B)}=0.$$

10. If $A+B+C=360°$, and if
$$\cos A=\frac{(d-a)(b-c)}{(d+a)(b+c)},\quad \cos B=\frac{(d-b)(c-a)}{(d+b)(c+a)},\quad \cos C=\frac{(d-c)(a-b)}{(d+c)(a+b)}$$
then
$$\tan\tfrac{1}{2}A+\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C=\pm 1.$$

11. Prove that
$$\tan\tfrac{1}{2}(x+y)\tan\tfrac{1}{2}(x-y)=\frac{\operatorname{cosec}2x\operatorname{cosec}y-\operatorname{cosec}2y\operatorname{cosec}x}{\operatorname{cosec}2x\operatorname{cosec}y+\operatorname{cosec}2y\operatorname{cosec}x}.$$

12. Shew that if $\cot\frac{1}{2}\alpha+\cot\frac{1}{2}\beta=2\cot\theta$, then

$$\{1-2\sec\theta\cos(\alpha-\theta)+\sec^2\theta\}\{1-2\sec\theta\cos(\beta-\theta)+\sec^2\theta\}=\tan^4\theta.$$

13. If $A+B+C+D=360°$, prove that

$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}D\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C-\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sin\tfrac{1}{2}A\sin\tfrac{1}{2}D$$
$$=\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D).$$

14. Prove that

$$\sin^2\tfrac{1}{2}(B-C)+\sin^2\tfrac{1}{2}(C-A)+\sin^2\tfrac{1}{2}(A-B)$$
$$+2\cos\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A-B)=2.$$

15. Prove that

$$\frac{\sin(y-z)+\sin(z-x)+\sin(x-y)}{1+\cos(y-z)+\cos(z-x)+\cos(x-y)}=-\tan\tfrac{1}{2}(y-z)\tan\tfrac{1}{2}(z-x)\tan\tfrac{1}{2}(x-y).$$

16. Investigate what relation must hold between α, β, γ, in order that

$$\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma.$$

17. If $A+B+C+D=360°$, prove that

$$\cos(B+C+D)+\cos(C+D+A)+\cos(D+A+B)+\cos(A+B+C)$$
$$=-4\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D).$$

18. If $\tan\frac{1}{2}\theta=\tan^3\frac{1}{2}\phi$, and $\tan\phi=2\tan\alpha$,

shew that $\theta+\phi=2\alpha$.

19. If $\sin^2\omega=\dfrac{\sin s\sin(s-\theta)\sin(s-\phi)\sin(s-\psi)}{4\cos^2\frac{1}{2}\theta\cos^2\frac{1}{2}\phi\cos^2\frac{1}{2}\psi}$, prove that

$$\tan^2\tfrac{1}{2}\omega=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-\theta)\tan\tfrac{1}{2}(s-\phi)\tan\tfrac{1}{2}(s-\psi),$$

where $2s=\theta+\phi+\psi$.

20. If $A+B+C+D=180°$, shew that

$$\sin A+\sin B+\sin C-\sin D=4\cos\tfrac{1}{2}(A+D)\cos\tfrac{1}{2}(B+D)\cos\tfrac{1}{2}(C+D).$$

21. If $\alpha+\beta+\gamma=2\pi$, prove that

$$\sin\beta(1+2\cos\gamma)+\sin\gamma(1+2\cos\alpha)+\sin\alpha(1+2\cos\beta)$$
$$=4\sin\tfrac{1}{2}(\gamma-\beta)\sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\alpha)$$

22. If $2s=a+b+c$, prove that

$$\cos\tfrac{1}{2}s\cos\tfrac{1}{2}(s-a)\cos\tfrac{1}{2}(s-b)\cos\tfrac{1}{2}(s-c)$$
$$+\sin\tfrac{1}{2}s\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)=\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c$$

23. If $\alpha+\beta+\gamma=\frac{1}{2}\pi$, then

$$\frac{(1-\tan\frac{1}{2}\alpha)(1-\tan\frac{1}{2}\beta)(1-\tan\frac{1}{2}\gamma)}{(1+\tan\frac{1}{2}\alpha)(1+\tan\frac{1}{2}\beta)(1+\tan\frac{1}{2}\gamma)}=\frac{\sin\alpha+\sin\beta+\sin\gamma-1}{\cos\alpha+\cos\beta+\cos\gamma}.$$

24. Prove that if $\alpha+\beta+\gamma=\pi$,

$$\cos(\tfrac{3}{2}\beta+\gamma-2\alpha)+\cos(\tfrac{3}{2}\gamma+\alpha-2\beta)+\cos(\tfrac{3}{2}\alpha+\beta-2\gamma)$$
$$=4\cos\tfrac{1}{4}(5\alpha-2\beta-\gamma)\cos\tfrac{1}{4}(5\beta-2\gamma-\alpha)\cos\tfrac{1}{4}(5\gamma-2\alpha-\beta)$$

25. If $\cos^2\theta=\cos\alpha/\cos\beta$, $\cos^2\theta'=\cos\alpha'/\cos\beta$,

and $\tan\theta/\tan\theta'=\tan\alpha/\tan\alpha'$,

shew that $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\alpha'=\pm\tan\frac{1}{2}\beta$.

26. If $\cos a = \cos\beta\cos\phi = \cos\beta'\cos\phi'$, and $\sin a = 2\sin\frac{1}{2}\phi\sin\frac{1}{2}\phi'$; shew that

$$\pm\tan\tfrac{1}{2}a = \tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\beta'.$$

27. If $A+B+C=180°$, and $\tan\frac{3}{4}A\tan\frac{3}{4}B = \tan\frac{3}{4}C$; shew that

$$\tan\tfrac{3}{4}A + \tan\tfrac{3}{4}B + \tan\tfrac{3}{4}C = \cot\tfrac{3}{4}A + \cot\tfrac{3}{4}B + \cot\tfrac{3}{4}C.$$

28. If

$$\tan\tfrac{1}{2}(y+z) + \tan\tfrac{1}{2}(z+x) + \tan\tfrac{1}{2}(x+y) = 0,$$

prove that

$$\sin x + \sin y + \sin z + 3\sin(x+y+z) = 0.$$

29. Prove that

$$\begin{aligned}\cos a\sin\tfrac{1}{2}(\theta+a)\sin\tfrac{1}{2}(\beta-\gamma) &+ \cos\beta\sin\tfrac{1}{2}(\theta+\beta)\sin\tfrac{1}{2}(\gamma-a)\\ &+ \cos\gamma\sin\tfrac{1}{2}(\theta+\gamma)\sin\tfrac{1}{2}(a-\beta)\\ &= 2\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-a)\sin\tfrac{1}{2}(a-\beta)\sin\tfrac{1}{2}(a+\beta+\gamma+\theta).\end{aligned}$$

30. Solve the equations

$$\left.\begin{aligned}\tan\tfrac{1}{2}a + \tan\tfrac{1}{2}\beta &= \tfrac{1}{3}\\ \tan a + \tan\beta &= \tfrac{3}{4}\end{aligned}\right\}.$$

31. If $\dfrac{\sin(\phi+a)\sin(\phi-a)}{\sin\left(\dfrac{a+\beta}{2}+2\theta\right)} = \dfrac{\sin(\phi+\beta)\sin(\phi-\beta)}{\sin\left(\dfrac{a+\beta}{2}-2\theta\right)} = \sin\tfrac{1}{2}(\beta-a)$;

shew that

$$\cos^2\tfrac{1}{2}a + \cos^2\tfrac{1}{2}\beta - \cos^2\theta = \tfrac{1}{2}.$$

32. If $\tan(\frac{1}{4}\pi+\frac{1}{2}\theta) = \tan^5(\frac{1}{4}\pi+\frac{1}{2}\phi)$, prove that

$$\sin\theta = 5\sin\phi\,\frac{(1+a^2\sin^2\phi)(1+\beta^2\sin^2\phi)}{(1+a^{-2}\sin^2\phi)(1+\beta^{-2}\sin^2\phi)};$$

and find a, β.

33. If $a+\beta+\gamma=\pi$, shew that

$$\begin{aligned}\tan^{-1}(\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma) + \tan^{-1}(\tan\tfrac{1}{2}\gamma\tan\tfrac{1}{2}a) + \tan^{-1}(\tan\tfrac{1}{2}a\tan\tfrac{1}{2}\beta)&\\ = \tan^{-1}\left\{1+\frac{8\sin\frac{1}{2}a\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin^2 a+\sin^2\beta+\sin^2\gamma}\right\}&.\end{aligned}$$

34. Prove that the sum of the three quantities

$$\frac{\cos^2\frac{1}{2}\gamma-\cos^2\frac{1}{2}\beta}{\cos^2\frac{1}{2}\beta\cos^2\frac{1}{2}\gamma+\sin^2\frac{1}{2}\beta\sin^2\frac{1}{2}\gamma},\quad \frac{\cos^2\frac{1}{2}a-\cos^2\frac{1}{2}\gamma}{\cos^2\frac{1}{2}a\cos^2\frac{1}{2}\gamma+\sin^2\frac{1}{2}a\sin^2\frac{1}{2}\gamma},$$

$$\frac{\cos^2\frac{1}{2}\beta-\cos^2\frac{1}{2}a}{\cos^2\frac{1}{2}\beta\cos^2\frac{1}{2}a+\sin^2\frac{1}{2}\beta\sin^2\frac{1}{2}a}$$

is equal to their continued product.

35. Prove that

$$\begin{aligned}\frac{\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)}+\frac{\cos\frac{1}{2}(\gamma+a)}{\cos\frac{1}{2}(\gamma-a)}+\frac{\cos\frac{1}{2}(a+\beta)}{\cos\frac{1}{2}(a-\beta)}-\frac{3\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\gamma+a)\cos\frac{1}{2}(a+\beta)}{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-a)\cos\frac{1}{2}(a-\beta)}&\\ =\frac{\cos a\cos\beta\cos\gamma-\cos(a+\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-a)\cos\frac{1}{2}(a-\beta)}&.\end{aligned}$$

36. Having given that

$$\frac{\cos a+\cos\beta+\cos\gamma}{\cos(a+\beta+\gamma)} = \frac{\sin a+\sin\beta+\sin\gamma}{\sin(a+\beta+\gamma)};$$

prove that each fraction is equal to

$$\cos(\beta+\gamma)+\cos(\gamma+a)+\cos(a+\beta),$$

and also to

$$\{\tan a-\tan\tfrac{1}{2}(\beta+\gamma)\}/\{\tan a+\tan\tfrac{1}{2}(\beta+\gamma)\}.$$

CHAPTER VI.

VARIOUS THEOREMS.

67. In this Chapter, we give various examples of transformations of expressions containing circular functions. Some of the theorems given are of intrinsic interest, others are given on account of the methods employed in proving them. Facility in the manipulation of expressions involving circular functions can only be obtained by much practice, but a careful study of the processes we employ in various cases will very materially assist the student in acquiring the power of dealing with this kind of symbols.

Identities and transformations.

68. EXAMPLES.

(1) *Prove that*

$$\sin 2\alpha \sin(\beta-\gamma)+\sin 2\beta \sin(\gamma-\alpha)+\sin 2\gamma \sin(\alpha-\beta)$$
$$=\{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\}\{\sin(\gamma-\beta)+\sin(\alpha-\gamma)+\sin(\beta-\alpha)\}.$$

The factors on the right-hand side of the equation are the sum and the difference respectively of the two quantities $\sin\gamma\cos\beta+\sin\alpha\cos\gamma+\sin\beta\cos\alpha$ and $\cos\gamma\sin\beta+\cos\alpha\sin\gamma+\cos\beta\sin\alpha$; hence the product of these factors is equal to

$$(\sin\gamma\cos\beta+\sin\alpha\cos\gamma+\sin\beta\cos\alpha)^2-(\cos\gamma\sin\beta+\cos\alpha\sin\gamma+\cos\beta\sin\alpha)^2.$$

Now $\sin^2\gamma\cos^2\beta-\cos^2\gamma\sin^2\beta=\sin^2\gamma-\sin^2\beta$, hence the algebraical sum of the square terms is zero; the product terms are equal to

$$2\sin\alpha\cos\alpha(\sin\beta\cos\gamma-\cos\beta\sin\gamma)+2\sin\beta\cos\beta(\sin\gamma\cos\alpha-\cos\gamma\sin\alpha)$$
$$+2\sin\gamma\cos\gamma(\sin\alpha\cos\beta-\cos\alpha\sin\beta),$$

and this is equal to

$$\sin 2\alpha\sin(\beta-\gamma)+\sin 2\beta\sin(\gamma-\alpha)+\sin 2\gamma\sin(\alpha-\beta);$$

thus the identity

$$\Sigma\sin 2\alpha\sin(\beta-\gamma)=\Sigma\sin(\beta+\gamma)\,\Sigma\sin(\gamma-\beta)$$

is proved.

(2) In the last example, put $\frac{1}{4}\pi+\alpha$, $\frac{1}{4}\pi+\beta$, $\frac{1}{4}\pi+\gamma$, for α, β, γ, respectively, we then obtain the identity

$$\Sigma \cos 2\alpha \sin(\beta-\gamma)=\Sigma \cos(\beta+\gamma)\,.\,\Sigma \sin(\gamma-\beta).$$

(3) *Prove that*

$$\Sigma \sin^3\alpha \sin(\beta-\gamma)=-\sin(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

In this case, as in many others, we replace the quantities $\sin^3\alpha$, $\sin^3\beta$, $\sin^3\gamma$, on the left-hand side of the equation, by the equivalent expressions in sines of multiple angles; the expression on the left-hand side then becomes

$$\tfrac{3}{4}\Sigma \sin\alpha \sin(\beta-\gamma)-\tfrac{1}{4}\Sigma \sin 3\alpha \sin(\beta-\gamma)$$

or $-\frac{1}{4}\Sigma \sin 3\alpha \sin(\beta-\gamma)$ in virtue of Ex. (3), Art. 45.

We now replace the products of sines by the difference of cosines, the expression then becomes

$$\tfrac{1}{8}\{\cos(3\alpha-\beta+\gamma)-\cos(3\alpha-\beta+\gamma)+\cos(3\beta+\gamma-\alpha)-\cos(3\beta-\gamma+\alpha)$$
$$+\cos(3\gamma+\alpha-\beta)-\cos(3\gamma-\alpha+\beta)\},$$

and the algebraic sum of the first and last terms in the bracket is

$$2\sin 2(\gamma-\alpha)\sin(\alpha+\beta+\gamma);$$

taking the second and third terms, and the fourth and fifth together, in the same way, the expression becomes

$$-\tfrac{1}{4}\sin(\alpha+\beta+\gamma)\,\Sigma \sin 2(\gamma-\alpha)$$

or $$-\sin(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$$

in virtue of Ex. (3), Art. 47.

(4) *Prove that*

$$\Sigma \cos^3\alpha \sin(\beta-\gamma)=\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

(5) *Prove that*

$$\Sigma \sin^3\alpha \sin^3(\beta-\gamma)=3\sin\alpha\sin\beta\sin\gamma\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta);$$

this follows from the fact that $x+y+z$ is a factor of $x^3+y^3+z^3-3xyz$; put $x=\sin\alpha\sin(\beta-\gamma)$, $y=\sin\beta\sin(\gamma-\alpha)$, $z=\sin\gamma\sin(\alpha-\beta)$, then $x+y+z=0$.

(6) *Prove that*

$$\sin(\alpha+\beta)\sin(\alpha-\beta)\sin(\gamma+\delta)\sin(\gamma-\delta)+\sin(\beta+\gamma)\sin(\beta-\gamma)\sin(\alpha+\delta)\sin(\alpha-\delta)$$
$$+\sin(\gamma+\alpha)\sin(\gamma-\alpha)\sin(\beta+\delta)\sin(\beta-\delta)=0.$$

The expression

$$(x^2-y^2)(z^2-w^2)+(y^2-z^2)(x^2-w^2)+(z^2-x^2)(y^2-w^2)$$

vanishes identically; put $x=\sin\alpha$, $y=\sin\beta$, $z=\sin\gamma$, $w=\sin\delta$, then remembering that

$$\sin^2\alpha-\sin^2\beta=\sin(\alpha+\beta)\sin(\alpha-\beta)$$

the theorem follows.

(7) *Prove that*

$$2(\cos\beta\cos\gamma-\cos\alpha)(\cos\gamma\cos\alpha-\cos\beta)(\cos\alpha\cos\beta-\cos\gamma)+\sin^2\alpha\sin^2\beta\sin^2\gamma$$
$$-\sin^2\alpha(\cos\beta\cos\gamma-\cos\alpha)^2-\sin^2\beta(\cos\gamma\cos\alpha-\cos\beta)^2-\sin^2\gamma(\cos\alpha\cos\beta-\cos\gamma)^2$$
$$=(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^2.$$

This follows from the known theorem that the square of the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ is equal to } \begin{vmatrix} bc-f^2 & fg-ch & fh-bg \\ fg-ch & ca-g^2 & gh-af \\ fh-bg & gh-af & ab-h^2 \end{vmatrix};$$

put $a=b=c=1$, $f=\cos\alpha$, $g=\cos\beta$, $h=\cos\gamma$, then $bc-f^2=\sin^2\alpha, \ldots$, expanding the determinant, the theorem follows.

(8) *Prove that*

$$\cos 2\alpha \cot\tfrac{1}{2}(\gamma-\alpha)\cot\tfrac{1}{2}(\alpha-\beta)+\cos 2\beta\cot\tfrac{1}{2}(\alpha-\beta)\cot\tfrac{1}{2}(\beta-\gamma)$$
$$+\cos 2\gamma\cot\tfrac{1}{2}(\beta-\gamma)\cot\tfrac{1}{2}(\gamma-\alpha)$$
$$=\cos 2\alpha+\cos 2\beta+\cos 2\gamma+2\cos(\beta+\gamma)+2\cos(\gamma+\alpha)+2\cos(\alpha+\beta).$$

Replace each cotangent on the left-hand side, by means of the formula $\cot\frac{1}{2}\theta=\dfrac{1+\cos\theta}{\sin\theta}$, then reduce the whole expression to the common denominator $\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$; the numerator becomes

$$\Sigma\cos 2\alpha\sin(\beta-\gamma)\{1+\cos(\gamma-\alpha)\}\{1+\cos(\alpha-\beta)\},$$

or $$\Sigma\cos 2\alpha\sin(\beta-\gamma)+\Sigma\cos 2\alpha\sin(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$$
$$+\Sigma\cos 2\alpha\sin(\beta-\gamma)\{\cos(\gamma-\alpha)+\cos(\alpha-\beta)\},$$

or $$\{1+\Sigma\cos(\beta-\gamma)\}\Sigma\cos 2\alpha\sin(\beta-\gamma)-\tfrac{1}{2}\Sigma\cos 2\alpha\sin 2(\beta-\gamma)$$
$$+\Sigma\cos 2\alpha\sin(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta).$$

Now $$1+\Sigma\cos(\beta-\gamma)=4\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta)$$
from Ex. 4, Art. 47,

and $$\Sigma\cos 2\alpha\sin(\beta-\gamma)=\Sigma\cos(\beta+\gamma)\,\Sigma\sin(\gamma-\beta)$$
$$=4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\,\Sigma\cos(\beta+\gamma).$$

Also $$\Sigma\cos 2\alpha\sin 2(\beta-\gamma)=0,$$

and $$\Sigma\cos 2\alpha\sin(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)=\tfrac{1}{4}\Sigma\cos 2\alpha\{\sin 2(\beta-\gamma)$$
$$-\sin 2(\gamma-\alpha)-\sin 2(\alpha-\beta)\}$$
$$=\tfrac{1}{2}\Sigma\cos 2\alpha\sin 2(\beta-\gamma)-\tfrac{1}{4}\Sigma\cos 2\alpha\,\Sigma\sin 2(\beta-\gamma),$$

which equals $$\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\,\Sigma\cos 2\alpha,$$

hence the numerator of the whole expression is equal to

$$\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\{2\Sigma\cos(\beta+\gamma)+\Sigma\cos 2\alpha\};$$

therefore the expression is equal to $2\Sigma\cos(\beta+\gamma)+\Sigma\cos 2\alpha$.

(9) *If*

$$\alpha+\beta+\gamma=\pi, \quad \textit{and} \quad \tan\tfrac{1}{4}(\beta+\gamma-\alpha)\tan\tfrac{1}{4}(\gamma+\alpha-\beta)\tan\tfrac{1}{4}(\alpha+\beta-\gamma)=1,$$

prove that $$1+\cos\alpha+\cos\beta+\cos\gamma=0.$$

Squaring the given equation, we have

$$\sin^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha)\sin^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\beta)\sin^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)$$
$$=\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha)\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\beta)\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma),$$

or $$(1-\sin\alpha)(1-\sin\beta)(1-\sin\gamma)=(1+\sin\alpha)(1+\sin\beta)(1+\sin\gamma),$$

hence $\sin\alpha+\sin\beta+\sin\gamma+\sin\alpha\sin\beta\sin\gamma=0,$
or $4\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma+\sin\alpha\sin\beta\sin\gamma=0;$
hence $1+2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma=0,$
also $\cos\alpha+\cos\beta+\cos\gamma-1=4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma;$
therefore $\cos\alpha+\cos\beta+\cos\gamma+1=0.$

(10) *Prove that if*

$$\tan\tfrac{1}{2}(\beta+\gamma-\alpha)\tan\tfrac{1}{2}(\gamma+\alpha-\beta)\tan\tfrac{1}{2}(\alpha+\beta-\gamma)=1,$$

then $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\cos\alpha\cos\beta\cos\gamma.$

We have

$$\sin\tfrac{1}{2}(\beta+\gamma-\alpha)\sin\tfrac{1}{2}(\gamma+\alpha-\beta)\sin\tfrac{1}{2}(\alpha+\beta-\gamma)$$
$$=\cos\tfrac{1}{2}(\beta+\gamma-\alpha)\cos\tfrac{1}{2}(\gamma+\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta-\gamma),$$

or $\{\cos(\beta-\alpha)-\cos\gamma\}\sin\tfrac{1}{2}(\alpha+\beta-\gamma)=\{\cos(\beta-\alpha)+\cos\gamma\}\cos\tfrac{1}{2}(\alpha+\beta-\gamma),$
which may be written

$$\cos(\beta-\alpha)\cos\tfrac{1}{2}(\alpha+\beta-\gamma+\tfrac{1}{2}\pi)+\cos\gamma\sin\tfrac{1}{2}(\alpha+\beta-\gamma+\tfrac{1}{2}\pi)=0.$$

Now $\sin 2\alpha+\sin 2\beta+\sin 2\gamma-4\cos\alpha\cos\beta\cos\gamma$ is equal to

$$2\sin(\alpha+\beta)\cos(\beta-\alpha)-2\cos\gamma\{\cos(\beta-\alpha)+\cos(\alpha+\beta)-\sin\gamma\},$$

or $2\cos(\beta-\alpha)\{\sin(\alpha+\beta)-\sin(\tfrac{1}{2}\pi-\gamma)\}-2\cos\gamma\{\cos(\beta+\alpha)-\cos(\tfrac{1}{2}\pi-\gamma)\},$
which is equal to

$$2\sin\tfrac{1}{2}(\alpha+\beta+\gamma-\tfrac{1}{2}\pi)\{\cos(\beta-\alpha)\cos\tfrac{1}{2}(\alpha+\beta-\gamma+\tfrac{1}{2}\pi)$$
$$+\cos\gamma\sin\tfrac{1}{2}(\alpha+\beta-\gamma+\tfrac{1}{2}\pi)\},$$

and this is equal to zero.

(11) *Having given that*

$$4\cos(y-z)\cos(z-x)\cos(x-y)=1,$$

prove that

$$1+12\cos 2(y-z)\cos 2(z-x)\cos 2(x-y)$$
$$=4\cos 3(y-z)\cos 3(z-x)\cos 3(x-y).$$

Let $\alpha=y-z,\quad \beta=z-x,\quad \gamma=x-y,$ then $\alpha+\beta+\gamma=0,$
hence $1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=0,$
therefore $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\tfrac{3}{2}.$

Now $\cos 3\alpha\cos 3\beta\cos 3\gamma=\cos\alpha\cos\beta\cos\gamma(4\cos^2\alpha-3)(4\cos^2\beta-3)(4\cos^2\gamma-3)$

$$=\tfrac{1}{4}(4-27-48\Sigma\cos^2\beta\cos^2\gamma+36\Sigma\cos^2\alpha)$$
$$=\tfrac{1}{4}(31-48\Sigma\cos^2\beta\cos^2\gamma)$$

and $\cos 2\alpha\cos 2\beta\cos 2\gamma=(2\cos^2\alpha-1)(2\cos^2\beta-1)(2\cos^2\gamma-1)$

$$=(\tfrac{1}{2}-1+3-4\Sigma\cos^2\beta\cos^2\gamma)$$
$$=\tfrac{5}{2}-4\Sigma\cos^2\beta\cos^2\gamma,$$

hence $4\cos 3\alpha\cos 3\beta\cos 3\gamma-12\cos 2\alpha\cos 2\beta\cos 2\gamma=1.$

(12) *Having given*

$$\frac{y^2+z^2-2yz\cos\alpha}{\sin^2\alpha}=\frac{z^2+x^2-2zx\cos\beta}{\sin^2\beta}=\frac{x^2+y^2-2xy\cos\gamma}{\sin^2\gamma},$$

prove that one of the following sets of equations holds[1], 2s *denoting* $\alpha+\beta+\gamma$;

$$\frac{x}{\cos(s-\alpha)}=\frac{y}{\cos(s-\beta)}=\frac{z}{\cos(s-\gamma)},$$

$$\frac{x}{\cos s}=\frac{y}{\cos(s-\gamma)}=\frac{z}{\cos(s-\beta)},$$

$$\frac{x}{\cos(s-\gamma)}=\frac{y}{\cos s}=\frac{z}{\cos(s-\alpha)},$$

$$\frac{x}{\cos(s-\beta)}=\frac{y}{\cos(s-\alpha)}=\frac{z}{\cos s}.$$

Let each of the equal fractions be denoted by k^2, and put $x=k\cos\theta$, $y=k\cos\phi$, $z=k\cos\psi$, we have then

$$\cos^2\phi+\cos^2\psi-2\cos\phi\cos\psi\cos\alpha=1-\cos^2\alpha,$$

or
$$(\cos\alpha-\cos\phi\cos\psi)^2=\sin^2\phi\sin^2\psi,$$

whence $\cos\alpha=\cos(\phi\pm\psi)$; similarly we can shew that $\cos\beta=\cos(\psi\pm\theta)$, $\cos\gamma=\cos(\theta\pm\phi)$, whence without loss of generality we can put $\alpha=\phi\pm\psi$, $\beta=\psi\pm\theta$, $\gamma=\theta\pm\phi$. In order that these equations may be consistent, we must take all the ambiguous signs to be positive, or else two of them negative and one positive. In the former case we find $\theta=s-\alpha$, $\phi=s-\beta$, $\psi=s-\gamma$; in the other cases we find the three sets of values

$$\left.\begin{array}{l}\theta=s\\ \phi=s-\gamma\\ \psi=\beta-s\end{array}\right\},\qquad \left.\begin{array}{l}\theta=\gamma-s\\ \phi=s\\ \psi=s-\beta\end{array}\right\},\qquad \left.\begin{array}{l}\theta=s-\beta\\ \phi=\alpha-s\\ \psi=s\end{array}\right\},$$

thus one of the four given relations is always satisfied.

The solution of equations.

69. EXAMPLES.

(1) *Solve the equation*

$$\sin 2\theta\sec 4\theta+\cos 2\theta=\cos 6\theta.$$

This equation may be written

$$\sin 2\theta\sec 4\theta+\cos 2\theta-\cos 6\theta=0,$$

or
$$\sin 2\theta\sec 4\theta+2\sin 4\theta\sin 2\theta=0;$$

hence
$$\sin 2\theta=0,\ \text{ or }\ \sec 4\theta+2\sin 4\theta=0,$$

that is
$$\sin 8\theta=-1.$$

Hence the solutions are

$$\theta=\tfrac{1}{2}m\pi,\quad \theta=\tfrac{1}{8}\left\{n\pi-(-1)^n\frac{\pi}{2}\right\}.$$

(2) *Solve the equation*[1]

$$\cos^3\alpha\sec x+\sin^3\alpha\operatorname{cosec} x=1,\ \textit{for } x.$$

We may write the equation

$$\cos^3\alpha\sin x+\sin^3\alpha\cos x=\sin x\cos x,$$

---

[1] This example is taken from Wolstenholme's problems.

or $$\sin^3 a\cos x-\cos a\sin^2 a\sin x=\sin x(\cos x-\cos a),$$

hence $$\sin^3 a\sin(a-x)=\sin x(\cos x-\cos a),$$

both sides are divisible by $\sin\frac{1}{2}(a-x)$, rejecting this factor, we have

$$2\sin^2 a\cos\tfrac{1}{2}(a-x)=2\sin x\sin\tfrac{1}{2}(a+x)=\cos\tfrac{1}{2}(x-a)-\cos\tfrac{1}{2}(3x+a),$$

therefore $$\cos\tfrac{1}{2}(3x+a)=\cos\tfrac{1}{2}(x-a)\cos 2a,$$

or $$2\cos\tfrac{1}{2}(3x+a)=\cos\tfrac{1}{2}(x+3a)+\cos\tfrac{1}{2}(x-5a),$$

which may be written

$$\cos\tfrac{1}{2}(3x+a)-\cos\tfrac{1}{2}(x+3a)=\cos\tfrac{1}{2}(x-5a)-\cos\tfrac{1}{2}(3x+a),$$

therefore $$\sin\tfrac{1}{2}(x-a)\sin(x+a)=-\sin(x-a)\sin\tfrac{1}{2}(x+3a);$$

again rejecting the factor $\sin\frac{1}{2}(x-a)$, we have

$$\sin(x+a)=-2\cos\tfrac{1}{2}(x-a)\sin\tfrac{1}{2}(x+3a)=-\{\sin(x+a)+\sin 2a\},$$

whence $$\sin(x+a)=-\sin a\cos a.$$

The solutions are therefore

$$x=2n\pi+a,\ \text{and}\ x=n\pi-a+(-1)^{n-1}\sin^{-1}(\sin a\cos a).$$

(3) *Solve the equations*

$$\left.\begin{aligned}a\sin(\mathrm{x}+\mathrm{y})-\mathrm{b}\sin(\mathrm{x}-\mathrm{y})&=2\mathrm{m}\cos\mathrm{x}\\ a\sin(\mathrm{x}+\mathrm{y})+\mathrm{b}\sin(\mathrm{x}-\mathrm{y})&=2\mathrm{n}\cos\mathrm{y}\end{aligned}\right\}.$$

We have

$$\frac{1}{n^2}\{a\sin(x+y)+b\sin(x-y)\}^2-\frac{1}{m^2}\{a\sin(x+y)-b\sin(x-y)\}^2$$
$$=4(\cos^2 y-\cos^2 x)=4\sin(x+y)\sin(x-y).$$

Let $\dfrac{\sin(x+y)}{\sin(x-y)}=t$, then t is given by the quadratic equation

$$a^2t^2\left(\frac{1}{n^2}-\frac{1}{m^2}\right)+2t\left\{ab\left(\frac{1}{n^2}+\frac{1}{m^2}\right)-2\right\}+b^2\left(\frac{1}{n^2}-\frac{1}{m^2}\right)=0.$$

Using t for either root of this equation, we have $t=\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}$, whence $\dfrac{\tan x}{\tan y}=\dfrac{t+1}{t-1}$; also dividing one of the given equations by the other, we have $\dfrac{m\cos x}{n\cos y}=\dfrac{at-b}{at+b}$; and thence eliminating y by means of these two equations and the relation $\sec^2 y-\tan^2 y=1$, we have

$$\frac{n^2}{m^2}\left(\frac{at-b}{at+b}\right)^2\sec^2 x-\left(\frac{t-1}{t+1}\right)^2\tan^2 x=1,$$

from which we find

$$\tan x=\pm\left\{1-\frac{n^2}{m^2}\left(\frac{at-b}{at+b}\right)^2\right\}^{\frac{1}{2}}\left\{\frac{n^2}{m^2}\left(\frac{at-b}{at+b}\right)^2-\left(\frac{t-1}{t+1}\right)^2\right\}^{-\frac{1}{2}},$$

which gives four values of $\tan x$, two corresponding to each root of the quadratic which determines t. Thus x is found, and then y is given by

$$\tan y=\frac{t-1}{t+1}\tan x.$$

Eliminations.

70. EXAMPLES.

(1) *Eliminate θ from the equations* $\dfrac{\cos^3\theta}{\cos(a-3\theta)} = \dfrac{\sin^3\theta}{\sin(a-3\theta)} = m.$

We have
$$m = \frac{\sin\theta\cos^3\theta + \cos\theta\sin^3\theta}{\sin(a-2\theta)} = \frac{\sin\theta\cos\theta}{\sin(a-2\theta)},$$

whence
$$\frac{1}{2m} = \sin a\cot 2\theta - \cos a.$$

Also
$$m = \frac{\cos^4\theta - \sin^4\theta}{\cos\theta\cos(a-3\theta) - \sin\theta\sin(a-3\theta)} = \frac{\cos 2\theta}{\cos(a-2\theta)}$$
$$= \frac{1}{\cos a + \sin a\tan 2\theta},$$

hence
$$\left(\frac{1}{2m} + \cos a\right)\left(\frac{1}{m} - \cos a\right) = \sin^2 a,$$

or
$$2m^2 - 1 = m\cos a,$$

the result of the elimination.

(2) *Shew that the result of eliminating θ from the equations*
$$\frac{\cos 3(\theta - a)}{\cos(\theta - \beta)} = \frac{\cos 3(\theta + a - \gamma)}{\cos(\theta + \beta - \gamma)} = \frac{\cos 3a}{\cos\beta}$$
is independent of β.

θ, $\gamma - \theta$, and zero are independent values of x which satisfy the equation
$$\frac{\cos 3(x-a)}{\cos(x-\beta)} = \frac{\cos 3a}{\cos\beta}.$$

We have
$$\cos 3x\cos 3a + \sin 3x\sin 3a = k(\cos x\cos\beta + \sin x\sin\beta),$$
where $k = \cos 3a/\cos\beta$; substituting for $\cos 3x$, $\sin 3x$ their values in terms of $\cos x$, $\sin x$ respectively, then dividing throughout by $\cos^3 x$, we have the following cubic in $\tan x$ $(=t)$,
$$\cos 3a\,\{4 - 3(1+t^2)\} + \sin 3a\,\{3t(1+t^2) - 4t^3\} = k(\cos\beta + t\sin\beta)(1+t^2)$$
or
$$t^3(k\sin\beta + \sin 3a) + t^2(k\cos\beta + 3\cos 3a) + t(k\sin\beta - 3\sin 3a)$$
$$+ k\cos\beta - \cos 3a = 0,$$
hence $\tan\theta$, and $\tan(\gamma - \theta)$, are the roots of the quadratic
$$t^2(k\sin\beta + \sin 3a) + t(k\cos\beta + 3\cos 3a) + k\sin\beta - 3\sin 3a = 0;$$

therefore
$$\tan\theta + \tan(\gamma-\theta) = -\frac{k\cos\beta + 3\cos 3a}{k\sin\beta + \sin 3a},$$

and
$$\tan\theta\tan(\gamma-\theta) = \frac{k\sin\beta - 3\sin 3a}{k\sin\beta + \sin 3a},$$

hence
$$\tan\gamma = \frac{-(k\cos\beta + 3\cos 3a)}{4\sin 3a} = -\cot 3a$$

or
$$\gamma - 3a = (2r+1)\tfrac{1}{2}\pi,$$

where r is any integer, thus the result of the elimination is independent of β.

(3) *Eliminate θ from the equations*

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad x\sin\theta-y\cos\theta=(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}.$$

Square each of the equations, and put $\tan\theta=t$, the equations become

$$t^2\left(1-\frac{y^2}{b^2}\right)-2t\frac{xy}{ab}+\left(1-\frac{x^2}{a^2}\right)=0,$$

$$t^2(a^2-x^2)+2txy+(b^2-y^2)=0,$$

respectively, and we have to eliminate t from them.

Solving for t^2 and t, we have

$$\frac{t^2}{2xy\left(1-\dfrac{x^2}{a^2}+\dfrac{b^2-y^2}{ab}\right)}=\frac{t}{\dfrac{(b^2-y^2)^2}{b^2}-\dfrac{(a^2-x^2)^2}{a^2}}=\frac{1}{-\dfrac{2xy(a^2-x^2)}{ab}-\dfrac{2xy(b^2-y^2)}{b^2}}$$

Hence

$$\left\{\frac{b^2-y^2}{b}+\frac{a^2-x^2}{a}\right\}^2\left\{\frac{b^2-y^2}{b}-\frac{a^2-x^2}{a}\right\}^2=-\frac{4x^2y^2}{a^3b^3}\{b(a^2-x^2)+a(b^2-y^2)\}^2$$

or
$$\left\{a+b-\frac{x^2}{a}-\frac{y^2}{b}\right\}^2\left\{\left(\frac{b^2-y^2}{b}-\frac{a^2-x^2}{a}\right)^2+\frac{4x^2y^2}{ab}\right\}=0,$$

hence
$$\frac{x^2}{a}+\frac{y^2}{b}=a+b$$

is the result of the elimination.

(4) *Eliminate θ from the equations*

$$x\sin\theta+y\cos\theta=2a\sin 2\theta,\quad x\cos\theta-y\sin\theta=a\cos 2\theta.$$

Solving for x and y, we find

$$x=a\cos\theta\,(2-\cos 2\theta),\quad y=a\sin\theta\,(2+\cos 2\theta)$$

or
$$x=a\cos\theta\,(\cos^2\theta+3\sin^2\theta),\quad y=a\sin\theta\,(3\cos^2\theta+\sin^2\theta),$$

therefore
$$x+y=a(\cos\theta+\sin\theta)^3,\quad x-y=a(\cos\theta-\sin\theta)^3,$$

hence
$$(x+y)^{\frac{2}{3}}=a^{\frac{2}{3}}(1+\sin 2\theta),\quad (x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}(1-\sin 2\theta)$$

and the result is

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

Relations between roots of equations.

71. EXAMPLES.

(1) *Consider the equation*

$$a\cos\theta+b\sin\theta=c.$$

Let α, β be distinct values of θ which satisfy it, then

$$a\cos\alpha+b\sin\alpha=c,$$

$$a\cos\beta+b\sin\beta=c;$$

therefore $$\frac{a}{\sin\beta-\sin\alpha}=\frac{b}{\cos\alpha-\cos\beta}=\frac{c}{\sin(\beta-\alpha)},$$

hence $$\tan\tfrac12(\beta+\alpha)=b/a,$$

and also $$\frac{1}{c}\cos\tfrac12(\beta-\alpha)=\frac{1}{b}\sin\tfrac12(\beta+\alpha)=\frac{1}{a}\cos\tfrac12(\beta+\alpha).$$

These relations may also be found as follows: put $\tan\frac12\theta=t$, then the given equation may be written

$$a(1-t^2)+2bt=c(1+t^2)$$

or $$t^2(c+a)-2bt+c-a=0.$$

The roots of this quadratic are $\tan\frac12\alpha$, $\tan\frac12\beta$,

hence $$\tan\tfrac12\alpha\tan\tfrac12\beta=\frac{c-a}{c+a},$$

whence we obtain the relation $\dfrac{\cos\frac12(\beta-\alpha)}{\cos\frac12(\beta+\alpha)}=\dfrac{c}{a}$.

Also $$\tan\tfrac12\alpha+\tan\tfrac12\beta=\frac{2b}{c+a},$$

from which the other relation may be obtained.

(2) *Consider the equation*

$$a\cos 2\theta+b\sin 2\theta+c\cos\theta+d\sin\theta+e=0.$$

Let $t=\tan\frac12\theta$, then the equation may be written as a biquadratic in t,

$$t^4(a-c+e)+t^3(-4b+2d)+t^2(-6a+2e)+t(4b+2d)+(a+c+e)=0;$$

if $$\tan\tfrac12\theta_1,\ \tan\tfrac12\theta_2,\ \tan\tfrac12\theta_3,\ \tan\tfrac12\theta_4$$

be the roots of this biquadratic, we have

$$\Sigma\tan\tfrac12\theta_1=\frac{4b-2d}{a-c+e},\quad \Sigma\tan\tfrac12\theta_1\tan\tfrac12\theta_2=\frac{2e-6a}{a-c+e},$$

$$\Sigma\tan\tfrac12\theta_1\tan\tfrac12\theta_2\tan\tfrac12\theta_3=-\frac{4b+2d}{a-c+e},\quad \tan\tfrac12\theta_1\tan\tfrac12\theta_2\tan\tfrac12\theta_3\tan\tfrac12\theta_4=\frac{a+c+e}{a-c+e},$$

and from these relations symmetrical functions of the four tangents may be calculated.

If $2s=\theta_1+\theta_2+\theta_3+\theta_4$ we have

$$\tan s=\frac{\Sigma\tan\frac12\theta_1-\Sigma\tan\frac12\theta_1\tan\frac12\theta_2\tan\frac12\theta_3}{1-\Sigma\tan\frac12\theta_1\tan\frac12\theta_2+\tan\frac12\theta_1\tan\frac12\theta_2\tan\frac12\theta_3\tan\frac12\theta_4}$$

$$=\frac{4b-2d+(4b+2d)}{a-c+e-(2e-6a)+a+c+e}=\frac{b}{a}.$$

We leave it as an exercise for the student to prove the relations

$$\frac{a}{\cos s}=\frac{b}{\sin s}=\frac{-c}{\Sigma\cos(s-\theta_1)}=\frac{-d}{\Sigma\sin(s-\theta_1)}=\frac{e}{\Sigma\cos\frac12(\theta_1+\theta_2-\theta_3-\theta_4)}.$$

(3) *If*

$$\sin\alpha\cos(\alpha+\theta)\tan 2\alpha=\sin\beta\cos(\beta+\theta)\tan 2\beta=\sin\gamma\cos(\gamma+\theta)\tan 2\gamma$$
$$=\sin\delta\cos(\delta+\theta)\tan 2\delta$$

and no two of the angles $\alpha, \beta, \gamma, \delta$ *differ by a multiple of* π, *shew that* $\alpha+\beta+\gamma+\delta+\theta$ *is a multiple of* π.

Write each of the equal quantities equal to k, then $\alpha, \beta, \gamma, \delta$ are roots of the equation

$$\sin x \cos (x+\theta) \tan 2x = k$$

which may be written

$$2 \tan^2 x (\cos\theta - \sin\theta \tan x) = k (1 - \tan^4 x),$$

hence $\quad \Sigma \tan\alpha = \dfrac{2\sin\theta}{k}, \quad \Sigma \tan\alpha \tan\beta = \dfrac{2\cos\theta}{k}, \quad \Sigma \tan\alpha\tan\beta\tan\gamma = 0,$

and $$\tan\alpha\tan\beta\tan\gamma\tan\delta = -1\,;$$

therefore $$\tan(\alpha+\beta+\gamma+\delta) = \frac{2\sin\theta}{k - 2\cos\theta - k} = -\tan\theta,$$

hence $\alpha+\beta+\gamma+\delta+\theta$ is a multiple of π.

(4) *If* α, β, γ *be unequal angles each less than* 2π, *prove that the equations*

$$\cos(\alpha+\theta)\sec 2\alpha = \cos(\theta+\beta)\sec 2\beta = \cos(\theta+\gamma)\sec 2\gamma$$

cannot coexist unless

$$\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0.$$

Writing k for each of the equal quantities we have

$$\cos\alpha\cos\theta - \sin\alpha\sin\theta - k\cos 2\alpha = 0,$$
$$\cos\beta\cos\theta - \sin\beta\sin\theta - k\cos 2\beta = 0,$$
$$\cos\gamma\cos\theta - \sin\gamma\sin\theta - k\cos 2\gamma = 0,$$

hence eliminating $\cos\theta$, $\sin\theta$, we have

$$\Sigma \cos 2\alpha \sin(\beta-\gamma) = 0$$

or $\quad \Sigma\cos(\beta+\gamma)\,\Sigma\sin(\gamma-\beta) = 0, \quad$ by Example (2), Art. 68,

hence $\quad \Sigma\cos(\beta+\gamma) = 0$ unless $\Sigma\sin(\gamma-\beta) = 0,$

that is unless $\quad \sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta) = 0.$

This example may also be solved in a similar manner to Example (3)

Maxima and minima. Inequalities.

72. EXAMPLES.

(1) *The greatest value of*

$$a\cos\theta + b\sin\theta \textit{ is } \sqrt{a^2+b^2}.$$

Put $b/a = \tan\alpha$, then $b = \sqrt{a^2+b^2}\sin\alpha$, $a = \sqrt{a^2+b^2}\cos\alpha$,

thus $$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\alpha),$$

now $\cos(\theta-\alpha)$ always lies between ± 1, hence $a\cos\theta + b\sin\theta$ lies between $\pm\sqrt{a^2+b^2}$.

(2) *If* $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, *then* u *lies between*

$$a+b \textit{ and } \sqrt{2(a^2+b^2)}.$$

Let $$x=a^2\cos^2\theta+b^2\sin^2\theta=\tfrac{1}{2}(a^2+b^2)+\tfrac{1}{2}(a^2-b^2)\cos 2\theta,$$

then $$u=\sqrt{x}+\sqrt{a^2+b^2-x},$$

$$u^2=a^2+b^2+2\sqrt{\tfrac{1}{4}(a^2+b^2)^2-\{\tfrac{1}{2}(a^2+b^2)-x\}^2},$$

hence u is greatest when $x=\frac{1}{2}(a^2+b^2)$, or the greatest value of u is $\sqrt{2(a^2+b^2)}$; also u is least when $\frac{1}{2}(a^2+b^2)-x$ is greatest, that is when x is least, which will be when $\cos 2\theta=-1$, in which case $x=b^2$, and then $u=a+b$; this therefore is the least value of u.

(3) *Shew that if θ lies between* 0 *and* π, $\cot\frac{1}{4}\theta-\cot\theta>2$.

We have

$$\cot\tfrac{1}{4}\theta-\cot\theta=\frac{\sin\frac{3}{4}\theta}{\sin\frac{1}{4}\theta\sin\theta}=\frac{3-4\sin^2\frac{1}{4}\theta}{\sin\theta}=\frac{1+2\cos\frac{1}{2}\theta}{\sin\theta},$$

hence $$\cot\tfrac{1}{4}\theta-\cot\theta=\operatorname{cosec}\theta+\operatorname{cosec}\tfrac{1}{2}\theta;$$

now $\operatorname{cosec}\theta$, $\operatorname{cosec}\frac{1}{2}\theta$ are each never less than unity, if θ lies between 0 and π, hence $\cot\frac{1}{4}\theta-\cot\theta>2$.

(4) *If the sum of* n *angles, each positive and less than $\frac{1}{2}\pi$, is given, shew that the sum or the product of the sines of the angles is greatest when the angles are all equal.*

A similar theorem holds for the cosines.

Let $a_1, a_2 \dots a_n$ be the angles and s be their sum. Then we have

$$\sin a_r+\sin a_s=2\sin\tfrac{1}{2}(a_r+a_s)\cos\tfrac{1}{2}(a_r-a_s),$$

now $\cos\frac{1}{2}(a_r-a_s)$ is less than unity unless $a_r=a_s$, hence

$$\sin a_r+\sin a_s<2\sin\tfrac{1}{2}(a_r+a_s)$$

unless $a_r=a_s$. If any two of the angles $a_1, a_2 \dots a_s$ are unequal, we can therefore increase $\Sigma\sin a$ by replacing each of those two angles by their arithmetical mean, hence $\Sigma\sin a$ is greatest when all the angles are equal: we have therefore $\Sigma\sin a \ngtr n\sin s/n$.

Again $$\sin a_r\sin a_s=\tfrac{1}{2}\{\cos(a_r-a_s)-\cos(a_r+a_s)\},$$

and this is less than $\frac{1}{2}\{1-\cos(a_r+a_s)\}$ or $\sin^2\frac{1}{2}(a_r+a_s)$ unless $a_r=a_s$. Hence as before, if any two angles in the product $\sin a_1$, $\sin a_2 \dots \sin a_n$ are unequal, we can make the product greater by replacing each of those two angles by the arithmetic mean of the two; it follows that $\sin a_1, \sin a_2 \dots \sin a_n$ is greatest when $a_1=a_2=\dots=a_n$, or the greatest value of the product is $(\sin s/n)^n$.

(5) *Under the same condition as in the last example, shew that the sum of the cosecants of the angles is least when the angles are all equal.*

We have

$$\operatorname{cosec}a_r+\operatorname{cosec}a_s$$
$$=\sin\tfrac{1}{2}(a_r+a_s)\left\{\frac{1}{\cos\frac{1}{2}(a_r-a_s)-\cos\frac{1}{2}(a_r+a_s)}+\frac{1}{\cos\frac{1}{2}(a_r-a_s)+\cos\frac{1}{2}(a_r+a_s)}\right\},$$

hence for a given value of a_r+a_s, $\operatorname{cosec}a_r+\operatorname{cosec}a_s$ has its least value when

$\cos \frac{1}{2}(\alpha_r - \alpha_s) = 1$, or when $\alpha_r = \alpha_s$. The reasoning is now similar to that in the last example.

(6) *Under the same conditions as in the last two examples, shew that the sum of the tangents or of the cotangents of the angles is least when the angles are all equal.*

(7) *Shew that if* $\alpha+\beta+\gamma=\pi$, $\cos\alpha\cos\beta\cos\gamma \not> 1/8$.

Porismatic systems of equations.

73. A system of equations is said to be porismatic[1] when the equations are inconsistent unless the coefficients satisfy a certain relation; when this relation is satisfied the equations have an infinite number of solutions.

The system

$$a\cos\beta\cos\gamma + b\sin\beta\sin\gamma + c + a'(\sin\beta+\sin\gamma) + b'(\cos\beta+\cos\gamma) + c'\sin(\beta+\gamma) = 0,$$

$$a\cos\gamma\cos\alpha + b\sin\gamma\sin\alpha + c + a'(\sin\gamma+\sin\alpha) + b'(\cos\gamma+\cos\alpha) + c'\sin(\gamma+\alpha) = 0,$$

$$a\cos\alpha\cos\beta + b\sin\alpha\sin\beta + c + a'(\sin\alpha+\sin\beta) + b'(\cos\alpha+\cos\beta) + c'\sin(\alpha+\beta) = 0,$$

is a system of three porismatic equations.

Consider the equation

$$a\cos\alpha\cos\theta + b\sin\alpha\sin\theta + c + a'(\sin\theta+\sin\alpha) + b'(\cos\theta+\cos\alpha) + c'\sin(\theta+\alpha) = 0,$$

this is satisfied by $\theta=\beta$, and by $\theta=\gamma$. Write this as an equation in $\tan\frac{1}{2}\theta = t$, thus:

$$t^2(-a\cos\alpha + c + a'\sin\alpha + b'\cos\alpha - b' - c'\sin\alpha) + 2t(b\sin\alpha + a' + c'\cos\alpha)$$
$$+ (a\cos\alpha + c + a'\sin\alpha + b' + b'\cos\alpha + c'\sin\alpha) = 0.$$

From this equation we find

$$\tan\tfrac{1}{2}\beta + \tan\tfrac{1}{2}\gamma, \text{ and } \tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma,$$

hence

$$\tan\tfrac{1}{2}(\beta+\gamma) = \frac{2(b\sin\alpha + a' + c'\cos\alpha)}{2(a\cos\alpha + b' + c'\sin\alpha)}.$$

We should find similarly

$$\tan\tfrac{1}{2}(\alpha+\gamma) = \frac{b\sin\beta + a' + c'\cos\beta}{a\cos\beta + b' + c'\sin\beta},$$

we can now deduce the value of $\tan\frac{1}{2}(\alpha-\beta)$; we find for the numerator the value

$$(b\sin\beta + a' + c'\cos\beta)(a\cos\alpha + b' + c'\sin\alpha) - (b\sin\alpha + a' + c'\cos\alpha)$$
$$(a\cos\beta + b' + c'\sin\beta)$$

or

$$2\sin\tfrac{1}{2}(\alpha-\beta)\{(c'^2 - ab)\cos\tfrac{1}{2}(\alpha-\beta) + (a'c' - bb')\cos\tfrac{1}{2}(\alpha+\beta)$$
$$- (aa' - b'c')\sin\tfrac{1}{2}(\alpha+\beta)\},$$

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[1] See *Proc. London Math. Soc.* Vol. IV. "On systems of Porismatic equations" by Wolstenholme.

and for the denominator,

$$(b\sin\alpha+a'+c'\cos\alpha)(b\sin\beta+a'+c'\cos\beta)+(a\cos\alpha+b'+c'\sin\alpha)(a\cos\beta+b'+c'\sin\beta)$$

or

$$(a^2+c'^2)\cos\alpha\cos\beta+(b^2+c'^2)\sin\alpha\sin\beta+(a'^2+b'^2)+(a'b+b'c')(\sin\alpha+\sin\beta)$$
$$+(a'c'+ab')(\cos\alpha+\cos\beta)+(a+b)c'\sin(\alpha+\beta);$$

dividing this fraction by $\sin\frac{1}{2}(\alpha-\beta)$, we have this denominator equal to

$$(c'^2-ab)\{1+\cos(\alpha-\beta)\}+(a'c'-bb')(\cos\alpha+\cos\beta)-(aa'-b'c')(\sin\alpha+\sin\beta),$$

hence

$$(a+b)\{a\cos\alpha\cos\beta+b\sin\alpha\sin\beta+c+a'(\sin\alpha+\sin\beta)+b'(\cos\alpha+\cos\beta)$$
$$+c'\sin(\alpha+\beta)\}$$

is equal to $$c'^2-a'^2-b'^2+ca+cb-ab.$$

Hence unless the condition

$$c'^2-a'^2-b'^2+ca+cb-ab=0$$

is satisfied, the system of equations cannot be satisfied except by equal values of α, β, γ. When this condition is satisfied, any one equation can be deduced from the other two.

The summation of series.

74. A large number of series involving circular functions can be summed by the method of differences. The most important example of the use of this method is the case of a series of sines or cosines of numbers in Arithmetical Progression.

Let the series be

$$S=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\},$$

we have $$\cos\alpha=\frac{1}{2\sin\frac{1}{2}\beta}\{\sin(\alpha+\tfrac{1}{2}\beta)-\sin(\alpha-\tfrac{1}{2}\beta)\},$$

$$\cos(\alpha+\beta)=\frac{1}{2\sin\frac{1}{2}\beta}\{\sin(\alpha+\tfrac{3}{2}\beta)-\sin(\alpha+\tfrac{1}{2}\beta)\},$$

....

$$\cos\{\alpha+(n-1)\beta\}$$
$$=\frac{1}{2\sin\frac{1}{2}\beta}\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)\right\};$$

whence $$S=\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\beta\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin(\alpha-\tfrac{1}{2}\beta)\right\}$$

$$=\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}\ldots\ldots(1).$$

In a similar manner we find

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}$$
$$= \sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2} \ldots\ldots(2).$$

The sum (2) may be deduced from (1) by changing α into $\alpha + \frac{1}{2}\pi$.

In (1) change β into $\beta+\pi$, we have then for the sum of the series

$$\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots + (-1)^{n-1}\cos\{\alpha+(n-1)\beta\},$$
$$\cos\left(\alpha+\frac{n-1}{2}\beta\right)\cos\frac{n\beta}{2}\sec\frac{\beta}{2}, \text{ or } \sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\sec\frac{\beta}{2},$$

according as n is odd or even. The sum of the series

$$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) \ldots$$

can be found from (2) in a similar manner.

EXAMPLES.

(1) *Prove that*

$$\sin na/\sin a = 2\{\cos(n-1)a + \cos(n-3)a + \cos(n-5)a + \ldots\},$$

and find a similar expansion for $\cos na/\cos a$.

(2) *Sum the series*

$$\cos^2 a + \cos^2(a+\beta) + \ldots + \cos^2\{a+(n-1)\beta\}.$$

We have

$$\cos^2 a = \tfrac{1}{2}(1+\cos 2a), \quad \cos^2(a+\beta) = \tfrac{1}{2}\{1+\cos 2(a+\beta)\}\ldots,$$

hence the sum required is

$$\tfrac{1}{2}n + \tfrac{1}{2}\cos\{2a+(n-1)\beta\}\sin n\beta\operatorname{cosec}\beta.$$

The sum of any positive integral powers of the terms of the series (1) and (2) may be found by a similar method.

(3) *Sum the series* $\operatorname{cosec} 2a + \operatorname{cosec} 2^2 a + \ldots + \operatorname{cosec} 2^n a$.

We find $\operatorname{cosec} 2a = \cot a - \cot 2a$, $\operatorname{cosec} 2^2 a = \cot 2a - \cot 2^2 a$,

$$\operatorname{cosec} 2^n a = \cot 2^{n-1} a - \cot 2^n a,$$

hence the sum required is $\cot a - \cot 2^n a$.

(4) *Sum the series*

$$\frac{3\sin x - \sin 3x}{\cos 3x} + \frac{3\sin 3x - \sin 3^2 x}{3\cos 3^2 x} + \ldots + \frac{3\sin 3^{n-1}x - \sin 3^n x}{3^{n-1}\cos 3^n x}.$$

We have $\tan 3^{n-1}x - \frac{1}{3}\tan 3^n x$

$$= \frac{3\sin 3^{n-1}x\cos 3^n x - \cos 3^{n-1}x\sin 3^n x}{3\cos 3^{n-1}x\cos 3^n x} = \frac{2\sin 3^{n-1}x\cos 3^n x - \sin 2\,.\,3^{n-1}x}{3\cos 3^{n-1}x\cos 3^n x}$$

$$=\frac{2\sin 3^{n-1}x\,(\cos 3^{n}x-\cos 3^{n-1}x)}{3\cos 3^{n-1}x\cos 3^{n}x}=\frac{-8\sin^{3}3^{n-1}x\cos 3^{n-1}x}{3\cos 3^{n-1}x\cos 3^{n}x}$$

$$=-2\frac{3\sin 3^{n-1}x-\sin 3^{n}x}{3\cos 3^{n}x},$$

whence $$\frac{3\sin x-\sin 3x}{\cos 3x}=\frac{3}{2}\left(\frac{1}{3}\tan 3x-\tan x\right),$$

hence $$\frac{3\sin 3x-\sin 3^{2}x}{3\cos 3^{2}x}=\frac{3}{2}\left(\frac{1}{3^{2}}\tan 3^{2}x-\frac{1}{3}\tan 3x\right),$$

……………………………………………………

$$\frac{3\sin 3^{n-1}x-\sin 3^{n}x}{3^{n-1}\cos 3^{n}x}=\frac{3}{2}\left(\frac{1}{3^{n}}\tan 3^{n}x-\frac{1}{3^{n-1}}\tan 3^{n-1}x\right);$$

therefore the sum of the series is

$$\frac{3}{2}\left(\frac{1}{3^{n}}\tan 3^{n}x-\tan x\right).$$

75. The sum of a series of either of the forms

$$u_1\cos\alpha+u_2\cos(\alpha+\beta)+u_3\cos(\alpha+2\beta)+\ldots+u_n\cos\{\alpha+(n-1)\beta\},$$

$$u_1\sin\alpha+u_2\sin(\alpha+\beta)+u_3\sin(\alpha+2\beta)+\ldots+u_n\sin\{\alpha+(n-1)\beta\},$$

can be found, if u_r is a rational integral function of r, of any positive integral degree s.

Let $S=u_1\cos\alpha+u_2\cos(\alpha+\beta)+\ldots+u_n\cos\{\alpha+(n-1)\beta\}$, then

$$2\cos\beta\,.\,S=u_1\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}+u_2\{\cos\alpha+\cos(\alpha+2\beta)\}$$
$$+\ldots+u_r\{\cos(\alpha+\overline{r-2}\beta)+\cos(\alpha+r\beta)\}$$
$$+\ldots+u_n\{\cos\{\alpha+\overline{n-2}\beta)+\cos(\alpha+n\beta)\},$$

whence

$$2(1-\cos\beta)S=(2u_1-u_2)\cos\alpha+(2u_2-u_1-u_3)\cos(\alpha+\beta)+\ldots$$
$$+(2u_r-u_{r-1}-u_{r+1})\cos(\alpha+\overline{r-1}\beta)$$
$$+\ldots+(2u_{n-1}-u_{n-2}-u_n)\cos(\alpha+\overline{n-2}\beta)$$
$$+(2u_n-u_{n-1})\cos(\alpha+n-1\beta)-u_1\cos(\alpha-\beta)-u_n\cos(\alpha+n\beta).$$

Now $2u_r-u_{r-1}-u_{r+1}$ is a rational integral function of r, of degree $s-1$, whence excluding the first and the three last terms, we have a series of the same kind, but of which the coefficients are of lower degree than in the given series. We again multiply by $1-\cos\beta$, and proceed in this way s times; the series will then be reduced to the form (1).

Examples.

(1) *Sum the series*

$$\cos\alpha+2\cos(\alpha+\beta)+3\cos(\alpha+2\beta)+\ldots+n\cos\{\alpha+(n-1)\beta\}.$$

We have in this case $2u_r-u_{r-1}-u_{r+1}=0$, $2u_1-u_2=0$, whence

$$2(1-\cos\beta)S=(n+1)\cos\{\alpha+(n-1)\beta\}-\cos(\alpha-\beta)-n\cos(\alpha+n\beta),$$

or $$S=\tfrac{1}{2}(n+1)\cos\{\alpha+(n-1)\beta\}/(1-\cos\beta)$$
$$-\tfrac{1}{2}\cos(\alpha-\beta)/(1-\cos\beta)-\tfrac{1}{2}n\cos(\alpha+n\beta)/(1-\cos\beta).$$

(2) *Sum the series*

$$\cos\alpha+2^2\cos(\alpha+\beta)+3^2\cos(\alpha+2\beta)+\ldots+n^2\cos\{\alpha+(n-1)\beta\}.$$

This series will be reduced to the last one by multiplication by $2(1-\cos\beta)$

76. The series

$$\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\ldots+x^{n-1}\cos\{\alpha+(n-1)\beta\},$$
$$\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\ldots+x^{n-1}\sin\{\alpha+(n-1)\beta\},$$

are recurring series of which the scale of relation is $1-2x\cos\beta+x^2$, for we have

$$\cos(\alpha+r\beta)+\cos(\alpha+\overline{r-2}\beta)=2\cos\beta\cos(\alpha+\overline{r-1}\beta),$$

and $$\sin(\alpha+r\beta)+\sin(\alpha+\overline{r-2}\beta)=2\cos\beta\sin(\alpha+\overline{r-1}\beta).$$

The series can therefore be summed by the ordinary rule for summing recurring series. If S denote the sum of the first series we find

$$S(1-2x\cos\beta+x^2)$$
$$=\cos\alpha-x\cos(\alpha-\beta)-x^n\cos(\alpha+n\beta)+x^{n+1}\cos\{\alpha+(n-1)\beta\}.$$

If $x<1$, we find, by making n indefinitely great, the limiting sum of the infinite series

$$\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\ldots$$

to be $\dfrac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$. Putting $\alpha=0$, we find

$$\frac{1-x\cos\beta}{1-2x\cos\beta+x^2}=1+x\cos\beta+x^2\cos2\beta+\ldots\ldots\text{ ad inf.},$$

whence also

$$\frac{1-x^2}{1-2x\cos\beta+x^2}=1+2x\cos\beta+2x^2\cos2\beta+\ldots\text{ ad inf.}\ldots\ldots(3).$$

77. In some cases the sum of a series may be found by means of a figure. We will take as an example the series (1) and (2) of Art. 74. Let OA_1, A_1A_2, A_2A_3, … $A_{n-1}A_n$ be equal chords of a circle, and let β be the angle between OA_1 produced and A_1A_2; draw a straight line OX so that $A_1OX=\alpha$, then the inclinations of $OA_1, A_1A_2, \ldots A_{n-1}A_n$, to OX, are $\alpha, \alpha+\beta, \alpha+2\beta, \ldots \alpha+(n-1)\beta$, and that of OA_n is $\alpha+\frac{1}{2}(n-1)\beta$; also if D be the diameter of the circle, we have

$$OA_1=D\sin\tfrac{1}{2}\beta, \quad OA_n=D\sin\tfrac{1}{2}n\beta.$$

Now the sum of the projections of $OA_1, A_1A_2, \ldots A_{n-1}A_n$, on OX, is

$$OA_1\cos\alpha+A_1A_2\cos(\alpha+\beta)+\ldots+A_{n-1}A_n\cos\{\alpha+(n-1)\beta\},$$

or $$D\sin\tfrac{1}{2}\beta\,[\cos\alpha+\cos(\alpha+\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}],$$

and this must equal the projection of OA_n which is

$$OA_n\cos\{\alpha+\tfrac{1}{2}(n-1)\beta\},$$

or $D\sin\frac{1}{2}n\beta\cos\{\alpha+\frac{1}{2}(n-1)\beta\}$, therefore

$$\cos\alpha+\cos(\alpha+\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}$$
$$=\cos\{\alpha+\tfrac{1}{2}(n-1)\beta\}\sin\tfrac{1}{2}n\beta\operatorname{cosec}\tfrac{1}{2}\beta.$$

If we project on a straight line perpendicular to OX we obtain the sum of the series of sines.

EXAMPLES.

(1) OA *is a diameter of a circle,* O, P, Q... *are points on the circumference such that each angle* PAO, QAP, RAQ... *is* α; AP, AQ, AR. . *meet the tangent at* O *in* p, q, r.... *Find by means of this figure the sum of the series*

$$\sec m\alpha\sec(m+1)\alpha+\sec(m+1)\alpha\sec(m+2)\alpha+\ldots \textit{ to } n \textit{ terms.}$$

(2) *Prove geometrically, that if* $\alpha, \beta, \gamma \ldots \kappa$ *be any number of angles,*

$$\sec\alpha\sec(\alpha+\beta)\sin\beta+\sec(\alpha+\beta)\sec(\alpha+\beta+\gamma)\sin\gamma$$
$$+\sec(\alpha+\beta+\gamma)\sec(\alpha+\beta+\gamma+\delta)\sin\delta+\ldots$$
$$=\sec\alpha\sec(\alpha+\beta+\gamma+\ldots+\kappa)\sin(\beta+\gamma+\ldots+\kappa).$$

EXAMPLES ON CHAPTER VI.

1 Eliminate θ from the equations

$$\cos^3\theta+a\cos\theta=b, \quad \sin^3\theta+a\sin\theta=c$$

2. Eliminate θ from the equations

$$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi), \quad a\cos 2\phi+b\cos 2\theta=c.$$

3. Prove that

$$\begin{aligned}&(a\sin\phi+b\cos\phi)(a\sin\psi+b\cos\psi)\sin(\phi-\psi)\\&\quad+(a\sin\psi+b\cos\psi)(a\sin\theta+b\cos\theta)\sin(\psi-\theta)\\&\quad+(a\sin\theta+b\cos\theta)(a\sin\phi+b\cos\phi)\sin(\theta-\phi)\\&\quad+(a^2+b^2)\sin(\phi-\psi)\sin(\psi-\theta)\sin(\theta-\phi)=0;\end{aligned}$$

and interpret the equation geometrically.

4. Reduce to its simplest form, and solve the equation

$$\cos^2\theta-\cos^2\alpha=2\cos^3\theta(\cos\theta-\cos\alpha)-2\sin^3\theta(\sin\theta-\sin\alpha).$$

5. Prove that the sum of three acute angles A, B, C, which satisfy the relation $\cos^2 A+\cos^2 B+\cos^2 C=1$, is less than 180°.

6. If $A+B+C=90°$, shew that the least value of $\tan^2 A+\tan^2 B+\tan^2 C$ is unity.

7. Find θ, ϕ from the equations

$$\left.\begin{aligned}\sin\theta+\sin\phi+\sin\alpha&=\cos\theta+\cos\phi+\cos\alpha\\\theta+\phi&=2\alpha\end{aligned}\right\}.$$

8. If $A+B+C=180°$, shew that $8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C \ngtr 1$.

9. If $\dfrac{x\sin\theta+y\sin\phi+z\sin\psi}{x\cos\theta+y\cos\phi+z\cos\psi}=\dfrac{4\sin\theta\sin\phi\sin\psi+\sin(\theta+\phi+\psi)}{4\cos\theta\cos\phi\cos\psi-\cos(\theta+\phi+\psi)}$,

prove that $\dfrac{x\sin\frac{1}{2}(\phi+\psi-\theta)+y\sin\frac{1}{2}(\psi+\theta-\phi)+z\sin\frac{1}{2}(\theta+\phi-\psi)}{x\cos\frac{1}{2}(\phi+\psi-\theta)+y\cos\frac{1}{2}(\psi+\theta-\phi)+z\cos\frac{1}{2}(\theta+\phi-\psi)}$

$$=\frac{4\sin\frac{1}{2}(\phi+\psi-\theta)\sin\frac{1}{2}(\psi+\theta-\phi)\sin\frac{1}{2}(\theta+\phi-\psi)+\sin\frac{1}{2}(\theta+\phi+\psi)}{4\cos\frac{1}{2}(\phi+\psi-\theta)\cos\frac{1}{2}(\psi+\theta-\phi)\cos\frac{1}{2}(\theta+\phi-\psi)-\cos\frac{1}{2}(\theta+\phi+\psi)}.$$

10. Prove that $\dfrac{\Sigma\sin 3\alpha\sin(\beta-\gamma)}{\Sigma\sin 2(\gamma-\beta)}=\sin(\alpha+\beta+\gamma)$,

and generally, if n be any odd number,

$$\frac{\Sigma\sin n\alpha\sin(\beta-\gamma)}{\Sigma\sin 2(\gamma-\beta)}=\Sigma\{\sin(p\alpha+q\beta+r\gamma)\},$$

where p, q, r are any odd numbers whose sum is n.

11. Having given

$$a^2\cos\alpha\cos\beta+a(\sin\alpha+\sin\beta)+1=0,$$

$$a^2\cos\alpha\cos\gamma+a(\sin\alpha+\sin\gamma)+1=0,$$

prove that $a^2\cos\beta\cos\gamma+a(\sin\beta+\sin\gamma)+1=0;$

β, γ being less than π.

12. If θ_1, θ_2 are the two values of θ which satisfy the equation

$$1+\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}=0,$$

shew that θ_1 and θ_2 being substituted for θ, ϕ in this equation will satisfy it

13. If

$$a\cos\alpha\cos\beta+b\sin\alpha\sin\beta=c,\quad a\cos\beta\cos\gamma+b\sin\beta\sin\gamma=c,$$

$$a\cos\gamma\cos\delta+b\sin\gamma\sin\delta=c,\quad a\cos\delta\cos\epsilon+b\sin\delta\sin\epsilon=c,$$

and $$a\cos\epsilon\cos\alpha+b\sin\epsilon\sin\alpha=c,$$

prove that $$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right),$$

the angles being all unequal and between 0 and 2π.

14. If

$$\sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta,\quad\text{and}\quad a\sin(\theta+\phi)+b\sin(\theta-\phi)=c,$$

prove that either

$$a\sin(2\alpha\pm2\beta)=-c,\quad\text{or}\quad a\sin2\alpha\pm b\sin2\beta=a.$$

15. If the equation

$$\sin^{2n+2}\theta/\sin^{2n}\alpha+\cos^{2n+2}\theta/\cos^{2n}\alpha=1$$

hold when $n=1$, shew that it will hold when n is any positive integer.

16. Eliminate θ from the equations

$$4(\cos\alpha\cos\theta+\cos\phi)(\cos\alpha\sin\theta+\sin\phi)$$
$$=4(\cos\alpha\cos\theta+\cos\psi)(\cos\alpha\sin\theta+\sin\psi)=(\cos\phi-\cos\psi)(\sin\phi-\sin\psi),$$

and prove that $\cos(\phi-\psi)=1$, or $\cos2\alpha$.

17. If $$\frac{\tan y}{\tan\beta}=\frac{\sin(x-\alpha)}{\sin\alpha}\quad\text{and}\quad\frac{\tan y}{\tan2\beta}=\frac{\sin(x-2\alpha)}{\sin2\alpha},$$

shew that $$\frac{\tan y}{\sin2\beta}=\frac{\sin x}{\sin2\alpha}=\frac{\cos x}{\cos2\alpha-\cos2\beta}.$$

18. Prove that the system of equations

$$\frac{\sin(2\alpha-\beta-\gamma)}{\cos(2\alpha+\beta+\gamma)}=\frac{\sin(2\beta-\gamma-\alpha)}{\cos(2\beta+\gamma+\alpha)}=\frac{\sin(2\gamma-\alpha-\beta)}{\cos(2\gamma+\alpha+\beta)},$$

if α, β, γ be unequal and each less than π, is equivalent to the single equation

$$\cos2(\beta+\gamma)+\cos2(\gamma+\alpha)+\cos2(\alpha+\beta)=0.$$

19. If

$$x=2\cos(\beta-\gamma)+\cos(\theta+\alpha)+\cos(\theta-\alpha)$$
$$=2\cos(\gamma-\alpha)+\cos(\theta+\beta)+\cos(\theta-\beta)$$
$$=-2\cos(\alpha-\beta)-\cos(\theta+\gamma)-\cos(\theta-\gamma),$$

prove that $x=\sin^2\theta$, if the difference between any two of the angles α, β, γ neither vanishes nor equals a multiple of π.

20. If $A+B+C=180°$ and if

$$\Sigma\sin(2n+1)A\sin(B-C)=0,$$

n being an integer, then shew that

$$\Sigma\sin(n-1)A\sin(n+1)(B-C)=0$$

21. If
$$\cot \tfrac{1}{2}(\alpha+\beta)(\cos\gamma-\cos\delta)+\cot\tfrac{1}{2}(\alpha+\gamma)(\cos\delta-\cos\beta)$$
$$+\cot\tfrac{1}{2}(\alpha+\delta)(\cos\beta-\cos\gamma)=0,$$
and no two of the angles are equal, or differ by a multiple of 2π, prove that
$$\cot\tfrac{1}{2}(\beta+\alpha)(\cos\gamma-\cos\delta)+\cot\tfrac{1}{2}(\beta+\gamma)(\cos\delta-\cos\alpha)$$
$$+\cot\tfrac{1}{2}(\beta+\delta)(\cos\alpha-\cos\gamma)=0.$$

22. If
$$\frac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)}+\frac{\sin(\beta+\theta)}{\sin(\beta+\phi)}=\frac{\cos(\alpha+\theta)}{\cos(\alpha+\phi)}+\frac{\cos(\beta+\theta)}{\cos(\beta+\phi)}=2,$$
shew that either α and β differ by an odd multiple of $\tfrac{1}{2}\pi$, or θ and ϕ differ by an even multiple of π.

23. If
$$a\cos(\phi+\psi)+b\cos(\phi-\psi)+c=0,$$
$$a\cos(\psi+\theta)+b\cos(\psi-\theta)+c=0,$$
$$a\cos(\theta+\phi)+b\cos(\theta-\phi)+c=0,$$
and if θ, ϕ, ψ are all unequal, shew that $a^2-b^2+2bc=0$.

24. If
$$\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta},$$
and β, γ are unequal, prove that each member will equal
$$\frac{\cos(\beta+\gamma+\theta)}{\sin(\beta+\gamma)\cos^2\alpha},$$
and
$$\cot\theta=\frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}.$$

25. If A, B, C be positive angles whose sum is 180°, prove that
$$\cos A+\cos B+\cos C>1 \text{ and } \not> 3/2.$$

26. Solve the equation
$$64\sin^7\theta+\sin 7\theta=0.$$

27. If $2s=x+y+z$, prove that
$$\tan(s-x)+\tan(s-y)+\tan(s-z)-\tan s$$
$$=\frac{4\sin x\sin y\sin z}{1-\cos^2 x-\cos^2 y-\cos^2 z-2\cos x\cos y\cos z},$$
$$\tan^{-1}(s-x)+\tan^{-1}(s-y)+\tan^{-1}(s-z)-\tan^{-1}s$$
$$=\tan^{-1}\frac{16xyz}{(x^2+y^2+z^2+4)^2-4(y^2z^2+z^2x^2+x^2y^2)}.$$

28. If
$$\frac{\cos\theta}{\cos\alpha}+\frac{\sin\theta}{\sin\alpha}=\frac{\cos\phi}{\cos\alpha}+\frac{\sin\phi}{\sin\alpha}=1,$$
prove that
$$\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}+1=0.$$

29. If
$$2\sin\alpha\cos(\theta+\phi)=2\cos(\theta-\phi)+\cos^2\alpha,$$
and
$$2\sin\alpha\cos(\theta+\psi)=2\cos(\psi-\theta)+\cos^2\alpha,$$
then
$$2\sin\alpha\cos(\phi+\psi)=2\cos(\phi-\psi)+\cos^2\alpha.$$

30. If $$\cos(y-z)+\cos(z-x)+\cos(x-y)=-3/2,$$
shew that
$$\cos^3(x+\theta)+\cos^3(y+\theta)+\cos^3(z+\theta)-3\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)=0,$$
for all values of θ.

31. If $$\frac{\sin r\alpha}{l}=\frac{\sin(r+1)\alpha}{m}=\frac{\sin(r+2)\alpha}{n},$$
prove that $$\frac{\cos r\alpha}{2m^2-l(l+n)}=\frac{\cos(r+1)\alpha}{m(n-l)}=\frac{\cos(r+2)\alpha}{n(l+n)-2m^2}.$$

32. Prove that the equations
$$\left(x+\frac{1}{x}\right)\sin\alpha=\frac{y}{z}+\frac{z}{y}+\cos^2\alpha,$$
$$\left(y+\frac{1}{y}\right)\sin\alpha=\frac{z}{x}+\frac{x}{z}+\cos^2\alpha,$$
$$\left(z+\frac{1}{z}\right)\sin\alpha=\frac{x}{y}+\frac{y}{x}+\cos^2\alpha,$$
are not independent, and that they are equivalent to
$$x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-\sin\alpha.$$

33. Prove that
$$2\cos(\beta-\gamma)\cos(\theta+\beta)\cos(\theta+\gamma)+2\cos(\gamma-\alpha)\cos(\theta+\gamma)\cos(\theta+\alpha)$$
$$+2\cos(\alpha-\beta)\cos(\theta+\alpha)\cos(\theta+\beta)-\cos 2(\theta+\alpha)-\cos 2(\theta+\beta)-\cos 2(\theta+\gamma)-1$$
is independent of θ, and exhibit its value as the product of cosines.

34. Prove that if α, β, γ, δ be four solutions of the equation
$$\tan(\theta+\tfrac{1}{4}\pi)=3\tan 3\theta,$$
no two of which have equal tangents, then
$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0,$$
$$\tan 2\alpha+\tan 2\beta+\tan 2\gamma+\tan 2\delta=4/3.$$

35. If $$6\tan(r+x)=3\tan(r+y)=2\tan(r+z),$$
shew that $$3\sin^2(x-y)+5\sin^2(y-z)-2\sin^2(z-x)=0.$$

36. Solve the equations
$$\left.\begin{aligned}\sin^{-1}x-\sin^{-1}y&=\tfrac{2}{3}\pi\\ \cos^{-1}x-\cos^{-1}y&=\tfrac{1}{3}\pi\end{aligned}\right\}.$$

37. Prove that the nth convergent to the continued fraction
$$\frac{1}{2\tan\alpha+}\,\frac{1}{2\tan\alpha+}\,\frac{1}{2\tan\alpha+}\cdots\cdots \text{ is } \frac{(\tan\alpha+\sec\alpha)^n-(\tan\alpha-\sec\alpha)^n}{(\tan\alpha+\sec\alpha)^{n+1}-(\tan\alpha-\sec\alpha)^{n+1}}.$$

38. Eliminate θ from the equations
$$\left.\begin{aligned}3a\cos\theta+a\cos 3\theta&=4x\\ 3a\sin\theta-a\sin 3\theta&=4y\end{aligned}\right\}$$

39. If $$\frac{\tan(\theta-\alpha)}{p}=\frac{\tan(\phi-\alpha)}{q}=\frac{\tan(\psi-\alpha)}{r},$$
prove that
$$p(q-r)^2\cot(\phi-\psi)+q(r-p)^2\cot(\psi-\theta)+r(p-q)^2\cot(\theta-\phi)=0.$$

40. Develop $$\frac{1}{1+a\cos\theta+b\sin\theta}$$
in a series of the form
$$A_0+A_1\cos(\theta-\alpha)+A_2\cos 2(\theta-\alpha)+\ldots.$$

41. Solve the equation
$$\tan 3\theta-\tan 2\theta-\tan\theta=0.$$

42. If
$$\cos^3 x+\cos^3 y=\cos 3\alpha,\quad \sin^3 x+\sin^3 y=\sin 3\alpha,\quad \text{and}\quad x+y=2\beta,$$
prove that $$8\sin^3 3(\alpha+\beta)=27\sin 2\beta\sin^2 4\beta\cos(3\alpha+\beta).$$

43. If
$$a\cos\phi\cos\psi+b\sin\phi\sin\psi=c,$$
$$a\cos\psi\cos\theta+b\sin\psi\sin\theta=c,$$
$$a\cos\theta\cos\phi+b\sin\theta\sin\phi=c,$$
prove that $$bc+ca+ab=0,\quad \text{unless}\quad a=b=c.$$

44. Solve the equation
$$\cos^{-1}(x+\tfrac{1}{2})+\cos^{-1}x+\cos^{-1}(x-\tfrac{1}{2})=\tfrac{3}{2}\pi.$$

45. Eliminate ϕ from the equations
$$a^3y\sin\phi+b^3x\cos\phi+ab(a^2\sin^2\phi+b)\cos^2\phi^2=0,$$
$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2.$$

46. Solve the equation
$$\cos 5\theta+5\cos 3\theta+10\cos\theta=\tfrac{1}{2}.$$

47. Eliminate θ from the equations
$$a\cos\theta\cos 2\theta=2(a\cos\theta-x),$$
$$a\sin\theta\sin 2\theta=2(a\sin\theta-y).$$

48. Prove that the number of solutions in positive integers (including zero), of the equation $3x+y=n$ (n integral), is
$$\tfrac{1}{3}\left[n+2+(-1)^n\frac{\cos\frac{1}{3}(2n+1)\pi}{\cos\frac{1}{3}\pi}\right].$$

49. Solve the equation
$$6\cos 3\theta-3\sin 3\theta-10\cos 2\theta+5\sin 2\theta+22\cos\theta-5\sin\theta=10.$$

50. Find the greatest value of
$$\frac{\operatorname{cosec}^2\theta-\tan^2\theta}{\cot^2\theta+\tan^2\theta-1}.$$

51. Prove that

$$\frac{\sec^2 a}{4} - \frac{\sec^2 a}{1} - \frac{\sec^2 a}{4} - \frac{\sec^2 a}{1} - \dots\dots$$

to r quotients is equal to

$$\frac{\sin ra}{2 \sin (r+1) a \cos a}.$$

52. Eliminate θ, ϕ from the equations

$$a \sin (\theta - a) + b \sin (\theta + a) = x \sin (\phi + \beta) + y \sin (\phi - \beta),$$
$$a \cos (\theta - a) - b \cos (\theta + a) = x \cos (\phi + \beta) - y \cos (\phi - \beta),$$
$$\theta \pm \phi = \gamma.$$

53 Prove that

$$\Sigma \cos a (\cos 3\beta - \cos 3\gamma)$$
$$= 4 (\cos \beta - \cos \gamma) (\cos \gamma - \cos a) (\cos a - \cos \beta) (\cos a + \cos \beta + \cos \gamma).$$

54. If
$$a \cos a + b \cos \beta + c \cos \gamma = 0,$$
$$a \sin a + b \sin \beta + c \sin \gamma = 0,$$
$$a \sec a + b \sec \beta + c \sec \gamma = 0,$$
prove that, in general, $\pm a \pm b \pm c = 0.$

55. Eliminate θ from the equations

$$\sin 3 (\tfrac{1}{4} \pi + \theta) + 3 \sin (\tfrac{1}{4} \pi + \theta) = 2a,$$
$$\sin 3 (\tfrac{1}{4} \pi - \theta) + 3 \sin (\tfrac{1}{4} \pi - \theta) = 2b.$$

56. If θ_1, θ_2, θ_3 be values of θ satisfying the equation $\tan (\theta + a) = k \tan 2\theta$, and such that no two of them differ by a multiple of π, prove that

$$\theta_1 + \theta_2 + \theta_3 + a$$

is a multiple of π.

57. Prove that

$$\Sigma \frac{\cos 4A}{\sin A \sin (A - B) \sin (A - C)} = 8 \sin (A + B + C) + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

58. Prove that

$$2 \{\sin^3 (\theta - a) \cos 2 (a - \phi) \sin (\beta - \gamma) + \sin^3 (\theta - \beta) \cos 2 (\beta - \phi) \sin (\gamma - a)$$
$$+ \sin^3 (\theta - \gamma) \cos 2 (\gamma - \phi) \sin (a - \beta)\}$$
$$= \{\sin 2a + \sin 2\beta + \sin 2\gamma - 3 \sin 2\theta\} \sin (\beta - \gamma) \sin (\gamma - a) \sin (a - \beta),$$

where $\phi = \frac{1}{2} (a + \beta + \gamma - 3\theta)$.

59. If $A + B + C + D = 180°$, prove that

$$(S - \sin A) (S - \sin B) (S - \sin C) (S - \sin D)$$
$$= \tfrac{1}{4} (\sin A \sin B + \sin C \sin D) (\sin B \sin C + \sin A \sin D) (\sin C \sin A + \sin B \sin D),$$

where $2S = \sin A + \sin B + \sin C + \sin D.$

60. Prove that the sum of the products of n terms of the series

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$$

taken two and two together is

$$\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}\beta\sec\tfrac{1}{2}\beta\sin\tfrac{1}{2}n\beta\left[\sin\tfrac{1}{2}n\beta\cos\tfrac{1}{2}\beta+\sin\tfrac{1}{2}(n-1)\beta\cos\{2\alpha+(n-1)\beta\}\right]-\tfrac{1}{4}n.$$

61. If $\dfrac{\cos\theta+\sin\theta}{2+\cos2\theta+\sin2\theta}=\dfrac{4(\cos\theta-\sin\theta)(\cos2\theta-\sin2\theta)}{4(\cos2\theta-\sin2\theta)^2-(\cos\theta-\sin\theta)^2}$,

shew that there will be three values of θ, such that

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=9.$$

62. If $\tan2\theta-\tan\theta=\tan2\phi-\tan\phi=\tan2\psi-\tan\psi$,

shew that $\theta+\phi+\psi$ is an odd multiple of $\frac{1}{2}\pi$, provided $\tan\theta$, $\tan\phi$, $\tan\psi$ are all unequal.

63. If

$$x\cos\alpha+y\sin\alpha+z+\cos2\alpha=0,$$
$$x\cos\beta+y\sin\beta+z+\cos2\beta=0,$$
$$x\cos\gamma+y\sin\gamma+z+\cos2\gamma=0,$$

prove that $x\cos\phi+y\sin\phi+z+\cos2\phi$

$$=8\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\phi)\sin\tfrac{1}{2}(\phi-\alpha)\sin\tfrac{1}{2}(\phi-\beta)\sin\tfrac{1}{2}(\phi-\gamma).$$

64. Eliminate θ, ϕ from the equations

$$\tan\theta+\tan\phi=a,$$
$$\sec\theta+\sec\phi=b,$$
$$\operatorname{cosec}\theta+\operatorname{cosec}\phi=c,$$

and shew that, if b and c are of the same sign, $bc>2a$.

65. Prove that the result of eliminating θ from the equations

$$\frac{\cos(\theta-3\alpha)}{\cos^3\alpha}=\frac{\cos(\theta-3\beta)}{\cos^3\beta}=\frac{\cos(\theta-3\gamma)}{\cos^3\gamma}$$

is $\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\{\cos(\alpha+\beta+\gamma)-4\cos\alpha\cos\beta\cos\gamma\}=0.$

66. If $(1-x+x^2)^{-1}$ be expanded in powers of x, shew that the coefficient of x^n is $\sin\frac{1}{3}(n+1)\pi/\sin\frac{1}{3}\pi$.

67. Prove that $\Sigma\cos4\alpha\sin(\beta+\gamma)\sin(\beta-\gamma)$

$$=-8\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$$

68. Prove that

$$\Sigma\cos2(\beta+\gamma-\alpha)\sin(\beta-\gamma)\cos\alpha=8\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\cos\alpha\cos\beta\cos\gamma.$$

69. If $a\sin\theta+b\cos\theta=a\operatorname{cosec}\theta+b\sec\theta$,

shew that each expression is equal to

$$(a^{\frac{2}{3}}-b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}.$$

70. Find the greatest value of

$$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta).$$

71. Solve the equation

$$\cos(x-a)\cos(x-b)\cos(x-c)=\sin a\sin b\sin c\sin x+\cos a\cos b\cos c\cos x$$

72. Solve the equation

$$\cos 2x+\cos 2(x-a)+\cos 2(x-b)+\cos 2(x-c)=4\cos a\cos b\cos a.$$

73. Solve the equation

$$\sin^3 3\alpha+\sin^3 2\alpha=\sin^2\alpha\,(\sin 3\alpha+\sin 2\alpha).$$

74. Eliminate θ from the equations

$$a\cos 2\theta+b\sin 2\theta=c,$$
$$a'\cos 3\theta+b'\sin 3\theta=0.$$

75. If $A+B+C=180°$, shew that

$$\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C+\sin^2\tfrac{1}{2}C\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}B$$

is not less than $\frac{1}{12}(\sin^2 A+\sin^2 B+\sin^2 C)$.

76. Eliminate θ from the equations

$$4x=5a\cos\theta-a\cos 5\theta$$
$$4y=5a\sin\theta-a\sin 5\theta$$

77. If $\cos 2\alpha\sin(\beta-\gamma)\sec(\beta+\gamma)$

$$=\cos 2\beta\sin(\gamma-\alpha)\sec(\gamma+\alpha)=\cos 2\gamma\sin(\alpha-\beta)\sec(\alpha+\beta),$$

prove that $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0,$

and $\sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta)=0.$

78. Prove that

$$\sum_{m=0}^{m=M}\cos(m\alpha+\beta)=\cos(\tfrac{1}{2}M\alpha+\beta)\sin\tfrac{1}{2}(M+1)\alpha\operatorname{cosec}\tfrac{1}{2}\alpha,$$

and $$\sum_{m=0}^{m=M}\sum_{n=0}^{n=N}\sum_{p=0}^{p=P}\ldots\ldots\cos(m\alpha+n\beta+p\gamma+\ldots\ldots)$$

$$=\cos\tfrac{1}{2}(M\alpha+N\beta+P\gamma+\ldots)\sin\tfrac{1}{2}(M+1)\alpha\sin\tfrac{1}{2}(N+1)\beta\ldots\operatorname{cosec}\tfrac{1}{2}\alpha\operatorname{cosec}\tfrac{1}{2}\beta\ldots.$$

Sum to n terms the following series in Exs. 79—93.

79 $\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\ldots\ldots+\sin^2 n\alpha.$

80. $\sin^2\alpha\sin 2\alpha+\sin^2 2\alpha\sin 3\alpha+\ldots\ldots+\sin^2 n\alpha\sin(n+1)\alpha.$

81. $\operatorname{cosec}\alpha\operatorname{cosec}(\alpha+\beta)$

$$+\operatorname{cosec}(\alpha+\beta)\operatorname{cosec}(\alpha+2\beta)+\ldots\ldots+\operatorname{cosec}\{\alpha+(n-1)\beta\}\operatorname{cosec}(\alpha+n\beta).$$

82. $\sin x\sin 2x\sin 3x$

$$+\sin 2x\sin 3x\sin 4x+\ldots\ldots+\sin nx\sin(n+1)x\sin(n+2)x.$$

83. $\sin^3\alpha+\frac{1}{3}\sin^3 3\alpha+\frac{1}{3^2}\sin^3 3^2\alpha+\ldots\ldots+\frac{1}{3^{n-1}}\sin^3 3^{n-1}\alpha.$

84. $\tan\theta\tan 3\theta+\tan 2\theta\tan 4\theta+\ldots\ldots+\tan n\theta\tan(n+2)\theta.$

85. $\tan\theta\sec 2\theta+\tan 2\theta\sec 2^2\theta+\ldots\ldots+\tan 2^{n-1}\theta\operatorname{sco} 2^n\theta.$

86. $\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\frac{x}{4}+\ldots\ldots+\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}.$

87. $\tan x\sec^2 x+\frac{1}{8}\tan\frac{x}{2}\sec^2\frac{x}{2}$

$$+\frac{1}{8^2}\tan\frac{x}{2^2}\sec^2\frac{x}{2^2}+\ldots\ldots+\frac{1}{8^{n-1}}\tan\frac{x}{2^{n-1}}\sec^2\frac{x}{2^{n-1}}.$$

88. $1+c\cos\theta\cos\phi+c^2\cos 2\theta\cos 2\phi+\ldots\ldots+c^{n-1}\cos(n-1)\theta\cos(n-1)\phi.$

89. $\frac{\cos 2\theta}{\sin^2 2\theta}+\frac{2\cos 4\theta}{\sin^2 4\theta}+\frac{4\cos 8\theta}{\sin^2 8\theta}+\ldots\ldots+\frac{2^{n-1}\cos 2^n\theta}{\sin^2 2^n\theta}.$

90. $\frac{\sin\theta}{\cos\theta+\cos 1^2\theta}+\frac{\sin 2\theta}{\cos 2\theta+\cos 2^2\theta}+\ldots\ldots+\frac{\sin n\theta}{\cos n\theta+\cos n^2\theta}.$

91. $\frac{\cot 2\alpha}{1-\cos^2 2\alpha\sec^2\alpha}+\frac{\cot 3\alpha}{1-\cos^2 3\alpha\sec^2\alpha}+\ldots\ldots+\frac{\cot(n+1)\alpha}{1-\cos^2(n+1)\alpha\,.\sec^2\alpha}$

92. $1\,.\,3\sin\frac{\pi}{n}+3\,.\,5\sin\frac{3\pi}{n}+\ldots\ldots+(2n-1)(2n+1)\sin\frac{(2n-1)\pi}{n}.$

93. $3\,.\,4\sin\alpha+4\,.\,5\sin 2\alpha+\ldots\ldots+(n+2)(n+3)\sin n\alpha.$

94. If θ_1, θ_2 be two solutions of the equation

$$\sin(\theta+\alpha)+\sin(\theta+\beta)+\sin(\alpha+\beta)=0,$$

where θ_1, θ_2, α, and β are each less than 2π,

shew that $$\sin(\theta_1+\theta_2)+\sin(\beta+\theta_1)+\sin(\beta+\theta_2)=0.$$

95. Prove that

$$\tfrac{1}{2}\cot^{-1}\frac{2\sqrt[3]{4}+1}{\sqrt{3}}+\tfrac{1}{3}\tan^{-1}\frac{\sqrt[3]{4}+1}{\sqrt{3}}=\tfrac{1}{6}\pi,$$

and $$\tfrac{1}{2}\tan^{-1}\frac{\sqrt[3]{2}+1}{\sqrt{3}}-\tfrac{1}{3}\tan^{-1}\frac{2\sqrt[3]{2}+1}{\sqrt{3}}=\tfrac{1}{36}\pi.$$

96. If α, β, γ, δ are four unequal values of θ, each less than 2π, which satisfy the equation

$$\cos 2(\lambda-\theta)+\cos(\mu-\theta)+\cos\nu=0,$$

prove that $$\alpha+\beta+\gamma+\delta-4\lambda=2n\pi,$$

and that $$\sin\tfrac{1}{2}(\beta+\gamma+\delta-\alpha-2\mu)+\sin\tfrac{1}{2}(\gamma+\delta+\alpha-\beta-2\mu)$$
$$+\sin\tfrac{1}{2}(\delta+\alpha+\beta-\gamma-2\mu)+\sin\tfrac{1}{2}(\alpha+\beta+\gamma-\delta-2\mu)=0.$$

CHAPTER VII.

EXPANSION OF FUNCTIONS OF MULTIPLE ANGLES.

Series in descending powers of the sine or cosine.

78. IF in the formula (40), of Art. 51, we write for $\sin^{2r} A$ its value $(1-\cos^2 A)^r$, and arrange the series in powers of $\cos A$, we shall obtain an expression for $\cos nA$ in powers of $\cos A$ only. Writing θ for A, we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(1-\cos^2\theta) + \dots$$
$$+ (-1)^r \frac{n(n-1)\dots(n-2r+1)}{(2r)!}\cos^{n-2r}\theta\,(1-\cos^2\theta)^r + \dots.$$

The coefficient of $(-1)^r\cos^{n-2r}\theta$ in this series is

$$\frac{n(n-1)\dots(n-2r+1)}{(2r)!} + \frac{n(n-1)\dots(n-2r-1)}{(2r+2)!}(r+1)$$
$$+ \frac{n(n-1)\dots(n-2r-3)}{(2r+4)!}\frac{(r+1)(r+2)}{2!} + \dots;$$

this is equal to the coefficient of x^{2r} in the product of $(1+x)^n$ and $(1-1/x^2)^{-(r+1)}$, x being supposed to be greater than unity; the coefficient is therefore equal to the coefficient of x^{r-1} in the expansion of $(1+x)^{n-r-1}(1-1/x)^{-(r+1)}$. This latter coefficient is equal to

$$\frac{(n-r-1)\dots(n-2r+1)}{r!}\left\{r + (n-2r)(r+1)\right.$$
$$\left. + \frac{(n-2r)(n-2r-1)}{2!}(r+2) + \dots\right\},$$

and this is equal to

$$\frac{(n-r-1)\dots(n-2r+1)}{r!}\{r(1+1)^{n-2r}+(n-2r)(1+1)^{n-2r-1}\},$$

or to

$$\frac{n(n-r-1)\dots(n-2r+1)}{r!}2^{n-2r-1}.$$

The coefficient of $\cos^n\theta$ is seen to be $\frac{1}{2}\{(1+1)^n+(1-1)^n\}$, or 2^{n-1}; the coefficient of $-\cos^{n-2}\theta$ is the term independent of x in the expansion of $(1+x)^{n-2}(1-1/x)^{-2}$, and this is easily seen to be $(1+1)^{n-2}+(n-2)(1+1)^{n-3}$, or $n\,.\,2^{n-3}$.

Hence we have

$$\cos n\theta = 2^{n-1}\cos^n\theta - \frac{n}{1!}2^{n-3}\cos^{n-2}\theta + \frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}\theta\dots(1),$$

of which the general term is

$$(-1)^r\frac{n(n-r-1)\dots(n-2r+1)}{r!}2^{n-2r-1}\cos^{n-2r}\theta.$$

In a similar manner we obtain from the formula (39) of Art. 51 the series

$$\sin n\theta/\sin\theta = 2^{n-1}\cos^{n-1}\theta - \frac{n-2}{1}2^{n-3}\cos^{n-3}\theta + \frac{(n-3)(n-4)}{2!}2^{n-5}\cos^{n-5}\theta - \dots\dots(2),$$

of which the general term is

$$(-1)^r\frac{(n-r-1)\dots(n-2r)}{r!}2^{n-2r-1}\cos^{n-2r-1}\theta.$$

79. If in the formulae (1) and (2) we change θ into $\frac{1}{2}\pi-\theta$, we obtain the formulae

$$(-1)^{\frac{n}{2}}\cos n\theta = 2^{n-1}\sin^n\theta - \frac{n}{1}2^{n-3}\sin^{n-2}\theta + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}\theta - \dots\dots(3),$$

$$(-1)^{\frac{n}{2}-1}\sin n\theta/\cos\theta = 2^{n-1}\sin^{n-1}\theta - \frac{n-2}{1}2^{n-3}\sin^{n-3}\theta + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}\theta - \dots\dots(4),$$

where n is even, and

$$(-1)^{\frac{1}{2}(n-1)}\sin n\theta = 2^{n-1}\sin^n\theta - \frac{n}{1}2^{n-3}\sin^{n-2}\theta + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}\theta - \dots\dots(5),$$

$$(-1)^{\frac{1}{2}(n-1)}\cos n\theta/\cos\theta = 2^{n-1}\sin^{n-1}\theta - \frac{n-2}{1}2^{n-3}\sin^{n-3}\theta + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}\theta - \ldots\ldots(6),$$

where n is odd.

Series in ascending powers of the sine or cosine.

80. In order to find expansions of $\cos n\theta$, $\sin n\theta$ in ascending powers of $\cos\theta$ or $\sin\theta$, we may write each of the six series we have obtained in the reverse order. It will, however, be better to obtain the required series directly.

First suppose n even, we have then

$$\cos n\theta = (1-\sin^2\theta)^{\frac{1}{2}n} - \frac{n(n-1)}{2!}(1-\sin^2\theta)^{\frac{1}{2}n-1}\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}(1-\sin^2\theta)^{\frac{1}{2}n-2}\sin^4\theta - \ldots\ldots;$$

expanding each power of $1-\sin^2\theta$ by the Binomial Theorem, we have

$$\cos n\theta = 1 - \left\{\frac{n}{2} + \frac{n(n-1)}{2}\right\}\sin^2\theta + \left\{\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2!} + \frac{n(n-1)}{2}\left(\frac{n}{2}-1\right) + \frac{n(n-1)(n-2)(n-3)}{4!}\right\}\sin^4\theta - \ldots\ldots\&c.,$$

the coefficient of $(-1)^s\sin^{2s}\theta$ being

$$\frac{\frac{1}{2}n(\frac{1}{2}n-1)\ldots(\frac{1}{2}n-s+1)}{s!} + \frac{n(n-1)}{2!}\frac{(\frac{1}{2}n-1)\ldots(\frac{1}{2}n-s+1)}{(s-1)!} + \frac{n(n-1)(n-2)(n-3)}{4!}\frac{(\frac{1}{2}n-2)\ldots(\frac{1}{2}n-s+1)}{(s-2)!} + \ldots\ldots,$$

which may be written in the form

$$\frac{1}{s!}\frac{n(n-2)(n-4)\ldots(n-2s+2)}{1.3.5\ldots(2s-1)}\left\{\left(\frac{2s-1}{2}\right)\left(\frac{2s-1}{2}-1\right)\ldots\left(\frac{2s-1}{2}-s+1\right) + s\left(\frac{2s-1}{2}\right)\left(\frac{2s-1}{2}-1\right)\ldots\left(\frac{2s-1}{2}-s+2\right)\left(\frac{n-1}{2}\right) + \frac{s(s-1)}{2!}\left(\frac{2s-1}{2}\right)\left(\frac{2s-1}{2}-1\right)\ldots\left(\frac{2s-1}{2}-s+3\right)\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right) + \ldots\ldots\right\}$$

Now, taking Vandermonde's theorem[1]

$$(p+q)_s = p_s + sp_{s-1}q_1 + \frac{s(s-1)}{2!}p_{s-2}q_2 + \dots,$$

where p_s denotes $p(p-1)\dots(p-s+1)$; since this holds for all values of p and q, let $p=\frac{2s-1}{2}$, $q=\frac{n-1}{2}$, then applying the theorem to the series in the brackets, we see that the coefficient of $(-1)^s \sin^{2s}\theta$ is

$$\frac{1}{s!}\,\frac{n(n-2)\dots(n-2s+2)}{1.3.5\dots(2s-1)}\left(\tfrac{1}{2}n+s-1\right)\left(\tfrac{1}{2}n+s-2\right)\dots\left(\tfrac{1}{2}n\right)$$

or

$$\frac{n^2(n^2-2^2)(n^2-4^2)\dots(n^2-\overline{2s-2}|^2)}{(2s)!}.$$

We have therefore, when n is even,

$$\cos n\theta = 1 - \frac{n^2}{2!}\sin^2\theta + \frac{n^2(n^2-2^2)}{4!}\sin^4\theta \dots$$
$$+ (-1)^s\frac{n^2(n^2-2^2)\dots(n^2-\overline{2s-2}|^2)}{(2s)!}\sin^{2s}\theta + \dots\dots(7);$$

this series is the series (3), written in the reverse order.

81. We have also

$$\sin n\theta = \cos\theta\left\{n(1-\sin^2\theta)^{\frac{1}{2}n-1}\sin\theta\right.$$
$$\left. - \frac{n(n-1)(n-2)}{3!}(1-\sin^2\theta)^{\frac{1}{2}n-2}\sin^3\theta + \dots\right\};$$

supposing n even, we expand each term of the series in powers of $\sin^2\theta$; we find the coefficient of $(-1)^{s+1}\cos\theta\sin^{2s-1}\theta$ to be

$$\frac{1}{(s-1)!}\,\frac{n(n-2)\dots(n-2s+2)}{1.3.5\dots(2s-1)}\left\{\left(\frac{2s-1}{2}\right)_{s-1} + (s-1)\left(\frac{2s-1}{2}\right)_{s-2}\left(\frac{n-1}{2}\right)_1\right.$$
$$+ \frac{(s-1)(s-2)}{2!}\left(\frac{2s-1}{2}\right)_{s-3}\left(\frac{n-1}{2}\right)_2$$
$$\left. + \dots\dots\dots\dots\dots\dots\dots\dots\right\}$$

which is equal to

$$\frac{1}{(s-1)!}\,\frac{n(n-2)\dots(n-2s+2)}{1.3.5\dots(2s-1)}\left(\tfrac{1}{2}n+s-1\right)\dots\left(\tfrac{1}{2}n+1\right)$$

or to

$$\frac{n(n^2-2^2)(n^2-4^2)\dots(n-\overline{2s-2}|^2)}{(2s-1)!}.$$

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[1] See Smith's *Algebra*, page 288.

We have therefore when n is even

$$\sin n\theta/\cos\theta = \frac{n}{1}\sin\theta - \frac{n(n^2-2^2)}{3\,!}\sin^3\theta + \dots$$
$$+ (-1)^{s-1}\frac{n(n^2-2^2)\dots(n^2-\overline{2s-2}|^2)}{(2s-1)!}\sin^{2s-1}\theta + \dots(8).$$

82. When n is odd, we have

$$\cos n\theta = \cos\theta\left\{(1-\sin^2\theta)^{\frac{1}{2}(n-1)} - \frac{n(n-1)}{2!}(1-\sin^2\theta)^{\frac{1}{2}(n-3)}\sin^2\theta + \dots\right\}$$

and $\sin n\theta = n(1-\sin^2\theta)^{\frac{1}{2}(n-1)}\sin\theta$

$$- \frac{n(n-1)(n-2)}{3\,!}(1-\sin^2\theta)^{\frac{1}{2}(n-3)}\sin^3\theta + \dots;$$

expanding in powers of $\sin\theta$, as in the last article, we find in a similar manner

$$\cos n\theta/\cos\theta = 1 - \frac{n^2-1^2}{2!}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4\,!}\sin^4\theta - \dots\dots$$
$$+ (-1)^s\frac{(n^2-1^2)(n^2-3^2)\dots(n^2-\overline{2s-1}|^2)}{(2s)\,!}\sin^{2s}\theta + \dots(9),$$

$$\sin n\theta = \frac{n}{1}\sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \dots\dots$$
$$+ (-1)^{s-1}\frac{n(n^2-1^2)\dots(n^2-\overline{2s-3}|^2)}{(2s-1)\,!}\sin^{2s-1}\theta + \dots(10).$$

83. If in the formulae (7), (8), (9), (10) we change θ into $\frac{1}{2}\pi - \theta$, we obtain the following formulae

$$(-1)^{\frac{1}{2}n}\cos n\theta = 1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2-2^2)}{4!}\cos^4\theta$$
$$- \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}\cos^6\theta + \dots(11),$$

$$(-1)^{\frac{1}{2}n+1}\sin n\theta/\sin\theta = \frac{n}{1}\cos\theta - \frac{n(n^2-2^2)}{3\,!}\cos^3\theta$$
$$+ \frac{n(n^2-2^2)(n^2-4^2)}{5\,!}\cos^5\theta - \dots(12),$$

when n is even, and

$$(-1)^{\frac{1}{2}(n-1)}\sin n\theta/\sin\theta = 1 - \frac{n^2-1^2}{2!}\cos^2\theta$$
$$+ \frac{(n^2-1^2)(n^2-3^2)}{4\,!}\cos^4\theta - \dots(13),$$

$$(-1)^{\frac{1}{2}(n-1)}\cos n\theta = \frac{n}{1}\cos\theta - \frac{n(n^2-1^2)}{3!}\cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^5\theta - \ldots (14),$$

when n is odd. These formulae are all the same as those of Arts. 78 and 79.

The circular functions of sub-multiple angles.

84. If in the formulae (1) to (6), or in the equivalent formulae (7) to (14), we write θ/n for θ, we obtain equations which give $\cos\frac{\theta}{n}$ or $\sin\frac{\theta}{n}$ when $\cos\theta$ and $\sin\theta$ are given. We will consider the various cases.

(1) Suppose $\cos\theta$ given, then the equation obtained from (1) will give us n values of $\cos\frac{\theta}{n}$. If $\cos\theta$ is given, we should expect to find the cosines of all the angles $\frac{2k\pi\pm\theta}{n}$, since $2k\pi\pm\theta$ represents all the angles which have the same cosine as θ, where k is any integer. Now whatever value k has, we can put $\pm k = s + k'n$, where s always has one of the values 0, 1, 2 ... $n-1$, and k' is a positive or negative integer. We have then

$$\cos\frac{2k\pi\pm\theta}{n} = \cos\left(\frac{\theta+2s\pi}{n} \pm 2\pi k'\right) = \cos\frac{\theta+2s\pi}{n},$$

thus we should expect to obtain the n values

$$\cos\frac{\theta}{n},\ \cos\frac{\theta+2\pi}{n},\ \cos\frac{\theta+4\pi}{n} \ldots\ldots \cos\frac{\theta+2(n-1)\pi}{n},$$

and these will be the roots of the equation we obtain from (1). These roots are in general all different, since neither the sum nor the difference of two of the angles is a multiple of 2π.

(2) Suppose $\cos\theta$ is given, then the equations obtained from (3) or (6) will give the values of $\sin\frac{\theta}{n}$. Before we use (6), we must square both sides and write $1-\sin^2\frac{\theta}{n}$ for $\cos^2\frac{\theta}{n}$; thus we obtain an equation of degree $2n$, for $\sin\frac{\theta}{n}$, when n is odd, and the equation

(3) gives us an equation of degree n when n is even. We expect to obtain all the values of $\sin\frac{2k\pi \pm \theta}{n}$ when $\cos\theta$ is given; as in the last case, we can shew that all these values are included in the expression $\sin\frac{2s\pi \pm \theta}{n}$, where s has the values $0, 1, 2 \ldots n-1$. When n is odd, all these values are different, and therefore we obtain $2n$ values which are the $2n$ roots of the equation obtained from (6). When n is even, we have $\sin\frac{(n-2s)\pi - \theta}{n} = \sin\frac{2s\pi + \theta}{n}$, hence in this case there are only n values, these being given by the equation obtained from (3).

(3) When $\sin\theta$ is given, we use the equation obtained from (2) to find $\cos\frac{\theta}{n}$: this gives $2n$ values of $\cos\frac{\theta}{n}$, for we must square both sides and replace $\sin^2\frac{\theta}{n}$ by $1-\cos^2\frac{\theta}{n}$, before using the equation. We shew as before that the expression $\cos\frac{s\pi + (-1)^s\theta}{n}$ has $2n$ values, so that we expect to find $\cos\frac{\theta}{n}$ given in terms of $\sin\theta$, by an equation of degree $2n$.

(4) If $\sin\theta$ is given, $\sin\frac{\theta}{n}$ will be given by (4) or (5), according as n is even or odd. When n is even, the equation from (4) gives $2n$ values of $\sin\frac{\theta}{n}$; these will be the $2n$ values of $\sin\frac{s\pi + (-1)^s\theta}{n}$. When n is odd, the equation formed from (5) gives n values of $\sin\frac{\theta}{n}$, these will be the n different values of $\sin\frac{s\pi + (-1)^s\theta}{n}$

Symmetrical functions of the roots of equations.

85. The formula (1) may be regarded as an equation of the nth degree in $\cos\theta$, when $\cos n\theta$ is given. Now each of the n angles $\theta, \theta + \frac{2\pi}{n}, \theta + \frac{4\pi}{n} \ldots\ldots \theta + \frac{2(n-1)\pi}{n}$, is such that the cosine of n

times the angle is equal to $\cos n\theta$, hence since $\cos\theta$, $\cos\left(\theta+\frac{2\pi}{n}\right)$, $\cos\left(\theta+\frac{4\pi}{n}\right)$...... $\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\}$ are all different, they are the n roots of the equation (1) in $\cos\theta$; we can now use the ordinary theorems for calculating symmetrical functions of the roots of equations to calculate symmetrical functions of the n cosines $\cos\left(\theta+\frac{2r}{n}\pi\right)$, r having the values $0, 1, 2 \ldots n-1$. We may of course, when it is convenient, use the forms (11) and (14) which are equivalent to (1). Again the equation (2) may be used to calculate symmetrical functions of the cosines of the $n-1$ angles for which $\sin n\theta/\sin\theta$ has a given value.

The equation (3) may be used in the same way to calculate symmetrical functions of the $2m$ sines

$$\sin\theta,\ \sin\left(\theta+\frac{\pi}{m}\right),\ \sin\left(\theta+\frac{2\pi}{m}\right)\ldots\ldots\sin\left(\theta+\frac{2m\pi-\pi}{m}\right),$$

where $n=2m$.

In the same way the theorem (5) may be used to calculat symmetrical functions of the $2m+1$ sines

$$\sin\theta,\ \sin\left(\theta+\frac{2\pi}{2m+1}\right),\ \sin\left(\theta+\frac{4\pi}{2m+1}\right)\ldots\ldots\sin\left(\theta+\frac{4m\pi}{2m+1}\right),$$

where $n=2m+1$.

The equation

$$\tan n\theta\left\{1-\frac{n(n-1)}{2!}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{4!}\tan^4\theta\ldots\ldots\right\}$$
$$=n\tan\theta-\frac{n(n-1)(n-2)}{3!}\tan^3\theta+\ldots\ldots$$

may be regarded as an equation in $\tan\theta$, of which the roots are

$$\tan\theta,\ \tan\left(\theta+\frac{\pi}{n}\right),\ \tan\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots\tan\left\{\theta+\frac{(n-1)\pi}{n}\right\},$$

and may therefore be used for calculating symmetrical functions of these expressions.

EXAMPLES.

(1) *Prove that the sum of the products of the cosecants of*

$$\theta,\ \theta+\frac{2\pi}{n} \dots\dots \theta+\frac{2(n-1)\pi}{n},$$

taken two at a time, is $-\frac{1}{4}n^2 \operatorname{cosec}^2 \frac{1}{2}n\theta$, *n being an even integer.*

Using the equation (7), the required sum is the sum of the products of the sines of the angles taken $n-2$ at a time divided by the product of all of them; this is equal to the coefficient of $\sin^2\theta$, divided by the term not involving $\sin\theta$, or $-\frac{n^2}{2(1-\cos n\theta)}$ which is equal to $-\frac{n^2}{4}\operatorname{cosec}^2\frac{1}{2}n\theta$.

(2) *Prove that*

$$\cos^4 \tfrac{1}{9}\pi + \cos^4 \tfrac{2}{9}\pi + \cos^4 \tfrac{3}{9}\pi + \cos^4 \tfrac{4}{9}\pi = 19/16$$

and

$$\sec^4 \tfrac{1}{9}\pi + \sec^4 \tfrac{2}{9}\pi + \sec^4 \tfrac{3}{9}\pi + \sec^4 \tfrac{4}{9}\pi = 1120.$$

If $\sin 9\theta/\sin\theta$ be expressed in terms of $\cos\theta$, and be then equated to zero, the values of $\cos\theta$ obtained by solving the equation of the eighth degree so obtained will be

$$\cos \tfrac{1}{9}\pi,\ \cos \tfrac{2}{9}\pi \dots\dots \cos \tfrac{8}{9}\pi.$$

We notice that

$$\cos \tfrac{8}{9}\pi = -\cos \tfrac{1}{9}\pi,\ \cos \tfrac{7}{9}\pi = -\cos \tfrac{2}{9}\pi \dots\dots,$$

thus

$$\pm\cos \tfrac{1}{9}\pi,\ \pm\cos \tfrac{2}{9}\pi,\ \pm\cos \tfrac{3}{9}\pi,\ \pm\cos \tfrac{4}{9}\pi$$

are the roots of the equation. We may either use the series (2), or proceed thus:—if $\sin 9\theta = 0$ we have

$$\sin 5\theta \cos 4\theta + \cos 5\theta \sin 4\theta = 0$$

or

$$(\sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta)(2\cos^2 2\theta - 1) + (\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta)\, 2 \sin 2\theta \cos 2\theta = 0;$$

substitute the values for $\sin 3\theta$, $\cos 2\theta$ ... and reject the factor $\sin\theta$, then let $x = \cos^2\theta$, we obtain the following biquadratic in x

$$\{(4x^2-1)(2x-1)+2(4x^2-3x)\}\{2(2x-1)^2-1\}+\{4(2x-1)(4x^2-3x) - 8(4x-1)(1-x)x\}(2x-1)=0$$

or

$$(16x^2-12x+1)(8x^2-8x+1)+(64x^3-80x^2+20x)(2x-1)=0$$

or, arranging according to powers of x,

$$256x^4 - 448x^3 + 240x^2 - 40x + 1 = 0.$$

The sum of the roots of this equation is 448/256, and the sum of the products of the roots taken two together is 240/256, hence the sum of the squares of the roots is $\frac{448^2 - 2 \,.\, 240 \,.\, 256}{(256)^2} = \frac{19}{16}$; also the sum of the squares of the reciprocals of the roots is $40^2 - 2 \,.\, 240$, or 1120.

(3) *Prove that* $\sin a + \sin 2a + \sin 4a = \frac{1}{2}\sqrt{7}$,

where $a = \frac{2}{7}\pi$.

We find $(\sin a + \sin 2a + \sin 4a)^2 = \sin^2 a + \sin^2 2a + \sin^2 4a.$

Now the roots of the equation $\sin 7\theta/\sin\theta = 0$ in $\sin\theta$ are

$$\pm\sin a, \quad \pm\sin 2a, \quad \pm\sin 4a;$$

put $x = \sin^2\theta$, then the equation in x is found to be

$$64x^3 - 112x^2 + 56x - 7 = 0,$$

hence $$\sin^2 a + \sin^2 2a + \sin^2 4a = 112/64 = 7/4;$$

therefore $$\sin a + \sin 2a + \sin 4a = \tfrac{1}{2}\sqrt{7}.$$

(4) *Evaluate* $\sin\dfrac{\pi}{17}$.

Writing $a = 2\pi/17$, we find by the formula for the sum of the cosines of angles in arithmetical progression

$$(\cos a + \cos 9a + \cos 13a + \cos 15a) + (\cos 3a + \cos 5a + \cos 7a + \cos 11a) = -\tfrac{1}{2}.$$

Also $(\cos a + \cos 9a + \cos 13a + \cos 15a)(\cos 3a + \cos 5a + \cos 7a + \cos 11a)$ is found, on multiplying out and replacing each product by half the sum of two cosines, to be equal to -1. The two quantities in brackets are therefore the roots of the quadratic $z^2 + \frac{1}{2}z - 1 = 0$, of which the roots are $\frac{1}{4}(-1 \pm \sqrt{17})$. It is easily seen that $\cos a + \cos 9a + \cos 13a + \cos 15a$ is positive, and

$$\cos 3a + \cos 5a + \cos 7a + \cos 11a$$

is negative, we have therefore

$$\cos a + \cos 9a + \cos 13a + \cos 15a = \tfrac{1}{4}(\sqrt{17} - 1),$$
$$\cos 3a + \cos 5a + \cos 7a + \cos 11a = -\tfrac{1}{4}(\sqrt{17} + 1).$$

We can now shew that $(\cos a + \cos 13a)(\cos 9a + \cos 15a) = -\frac{1}{4}$, hence $\cos a + \cos 13a$, $\cos 9a + \cos 15a$ are the roots of the quadratic

$$x^2 - \tfrac{1}{4}(\sqrt{17} - 1)x - \tfrac{1}{4} = 0,$$

hence $$\cos a + \cos 13a = \tfrac{1}{8}\left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right);$$

similarly we find $$\cos 3a + \cos 5a = \tfrac{1}{8}\left(-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}\right).$$

Now $\cos a \cos 13a = \frac{1}{2}(\cos 12a + \cos 14a) = \frac{1}{2}(\cos 3a + \cos 5a)$; and since we have thus found the sum and the product of $\cos a$, $\cos 13a$, we can find each of them. Noticing that $\cos a > \cos 13a$, we have

$$\cos a = \tfrac{1}{16}\left\{\sqrt{17} - 1 + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}\right\}.$$

We have then

$$\sin \pi/17 = \sqrt{\tfrac{1}{2}(1 - \cos a)}$$
$$= \tfrac{1}{8}\sqrt{34 - 2\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}} - 4\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}}.$$

(5) *Shew*[1] *that, if* f(x, y) *be a homogeneous function of* x, y *of* $n-1$ *dimensions,*

$$\frac{f(\sin x, \cos x)}{\sin(x - a_1)\sin(x - a_2)\ldots\sin(x - a_n)}$$
$$= \sum_{r=1}^{r=n} \frac{f(\sin a_r, \cos a_r)}{\sin(x - a_r)\sin(a_r - a_1)\sin(a_r - a_2)\ldots\sin(a_r - a_n)}.$$

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[1] This theorem was given by Hermite in a memoir "Sur l'Intégration des Fonctions circulaires" in the *Proc. Lond. Math Soc.* for 1872.

The expression on the left-hand side of the equation may be written

$$\frac{f(t,1)}{(t-a_1)(t-a_2)\ldots(t-a_n)}\cdot\frac{1}{\cos x\cos a_1\cos a_2\ldots\cos a_n},\ \text{where } t=\tan x,\ a_r=\tan a_r.$$

Now since $f(t, 1)$ is of degree $n-1$, lower than n, we have by the ordinary method of resolving into partial fractions

$$\frac{f(t,1)}{(t-a_1)(t-a_2)\ldots(t-a_n)}=\sum_{r=1}^{r=n}\frac{f(a_r,1)}{(t-a_r)(a_r-a_1)(a_r-a_2)\ldots(a_r-a_n)}$$

$$=\Sigma\frac{f(\sin a_r,\cos a_r).\cos x\cos a_1\cos a_2\ldots\cos a_n}{\sin(x-a_r)\sin(a_r-a_1)\ldots\sin(a_r-a_n)},$$

thus the result follows.

Factorization.

86. Since $\cos n\theta$ can be expressed as a rational integral function of the nth degree in $\cos\theta$, we can express $\cos n\theta$ as the product of n factors linear in $\cos\theta$; the values of $\cos\theta$ for which $\cos n\theta$ vanishes are

$$\cos\frac{\pi}{2n},\ \cos\frac{3\pi}{2n}\ldots\ldots\cos\frac{(2n-1)\pi}{2n};$$

these cosines are all different; therefore

$$\cos n\theta=A\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\ldots\ldots\left(\cos\theta-\cos\frac{(2n-1)\pi}{2n}\right),$$

where A is a numerical factor. Since the highest power of $\cos\theta$ in the expression for $\cos n\theta$ is $2^{n-1}\cos^n\theta$, we see that $A=2^{n-1}$; therefore

$$\cos n\theta=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\ldots\ldots\left(\cos\theta-\cos\frac{(2n-1)\pi}{2n}\right).$$

Now $\cos\frac{r\pi}{2n}=-\cos\frac{(2n-r)\pi}{2n}$, therefore this expression may be written

$$\cos n\theta=2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{2n}\right)\ldots\ldots\left(\cos^2\theta-\cos^2\frac{(n-2)\pi}{2n}\right)\cos\theta,$$

when n is odd, and

$$\cos n\theta = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta - \cos^2\frac{3\pi}{2n}\right)\dots\dots\left(\cos^2\theta - \cos^2\frac{(n-1)\pi}{2n}\right),$$

when n is even; these expressions may also be written

$$\cos n\theta/\cos\theta = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\dots\dots\left(\sin^2\frac{(n-2)\pi}{2n} - \sin^2\theta\right),$$

when n is odd, and

$$\cos n\theta = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\dots\dots\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\theta\right),$$

when n is even.

In each of these equations put $\theta = 0$, we then obtain the theorems

$$\left.\begin{array}{l} 2^{\frac{1}{2}(n-1)}\sin\dfrac{\pi}{2n}\sin\dfrac{3\pi}{2n}\dots\dots\sin\dfrac{(n-2)\pi}{2n} = 1, \\ \text{when } n \text{ is odd, and} \\ 2^{\frac{1}{2}(n-1)}\sin\dfrac{\pi}{2n}\sin\dfrac{3\pi}{2n}\dots\dots\sin\dfrac{(n-1)\pi}{2n} = 1, \end{array}\right\}\dots(15),$$

when n is even.

The positive sign is taken in extracting the square root, since the angles are all acute.

If we divide the expressions for $\cos n\theta/\cos\theta$ or $\cos n\theta$ by the corresponding one of the products in (15) squared, we obtain the expressions

$$\frac{\cos n\theta}{\cos\theta} = \left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\dots\dots\left(1 - \frac{\sin^2\theta}{\sin^2\frac{(n-2)\pi}{2n}}\right)\dots(16),$$

when n is odd, and

$$\cos n\theta = \left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\dots\dots\left(1 - \frac{\sin^2\theta}{\sin^2\frac{(n-1)\pi}{2n}}\right)\dots(17),$$

when n is even.

We may write the theorems (16) and (17) thus:—

$$\cos n\theta/\cos\theta = \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\sin^2\theta}{\sin^2\frac{(2r-1)\pi}{2n}}\right) \quad\ldots\ldots(16),$$

where n is odd, and

$$\cos n\theta = \prod_{r=1}^{r=\frac{1}{2}n} \left(1 - \frac{\sin^2\theta}{\sin^2\frac{(2r-1)\pi}{2n}}\right) \quad\ldots\ldots\ldots(17),$$

where n is even.

87. As in the last article, since $\sin n\theta/\sin\theta$ is an algebraical function of degree $n-1$ in $\cos\theta$, we may find a corresponding expression for it in factors linear in $\cos\theta$; in this case

$$\cos\frac{\pi}{n},\quad \cos\frac{2\pi}{n}\ldots\cos\frac{(n-1)\pi}{n}$$

are the values of $\cos\theta$ for which $\sin n\theta/\sin\theta$ is equal to zero. These values may be thus grouped $\pm\cos\frac{\pi}{n}$, $\pm\cos\frac{2\pi}{n}\ldots\ldots$; hence as before

$$\sin n\theta/\sin\theta = 2^{n-1}\cos\theta\left(\cos^2\theta - \cos^2\frac{\pi}{n}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{n}\right)\ldots$$
$$\left(\cos^2\theta - \cos^2\frac{(n-2)\pi}{2n}\right),$$

when n is even, and

$$\sin n\theta/\sin\theta = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{n}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{n}\right)\ldots$$
$$\left(\cos^2\theta - \cos^2\frac{(n-1)\pi}{2n}\right),$$

when n is odd.

We can write these equations in the forms

$$\sin n\theta/\sin\theta = 2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots$$
$$\left(\sin^2\frac{(n-2)\pi}{2n} - \sin^2\theta\right),$$

when n is even, and

$$\sin n\theta/\sin\theta = 2^{n-1}\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots$$
$$\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\theta\right),$$

when n is odd.

We shall shew in the next Chapter that $\sin n\theta/\sin\theta$ has the limit n when θ is indefinitely diminished; hence

$$\sqrt{n} = 2^{\frac{n-1}{2}} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \text{..................}(18),$$

the last factor being $\sin\frac{(n-2)\pi}{2n}$ or $\sin\frac{(n-1)\pi}{2n}$, according as n is even or odd. Hence

$$\sin n\theta/n\sin\theta = \cos\theta \prod_{r=1}^{r=\frac{1}{2}(n-2)} \left(1 - \frac{\sin^2\theta}{\sin^2\frac{r\pi}{n}}\right) \text{......}(19),$$

when n is even, and

$$\sin n\theta/n\sin\theta = \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\sin^2\theta}{\sin^2\frac{r\pi}{n}}\right) \text{.........}(20),$$

when n is odd.

88. The expression $\cos n\theta - \cos n\phi$ may be regarded as an algebraical function of $\cos\theta$ of degree n, and can therefore be factorised; the values of $\cos\theta$ for which the expression vanishes are $\cos\phi$, $\cos\left(\phi + \frac{2\pi}{n}\right)$, $\cos\left(\phi + \frac{4\pi}{n}\right)$......, hence

$$\cos n\theta - \cos n\phi = 2^{n-1} \prod_{r=0}^{r=n-1} \left\{\cos\theta - \cos\left(\phi + \frac{2r\pi}{n}\right)\right\} \text{...}(21).$$

89. [1] We shall now factorise the expression $x^{2n} - 2x^n \cos n\theta + 1$. We have

$$\begin{aligned} x^n - 2\cos n\theta + x^{-n} = {} & (x^{n-1} + x^{-n+1})(x - 2\cos\theta + x^{-1}) \\ & + 2\cos\theta\,(x^{n-1} - 2\cos(n-1)\,\theta + x^{-n+1}) \\ & - (x^{n-2} - 2\cos(n-2)\,\theta + x^{-n+2}). \end{aligned}$$

If we denote $x^n - 2\cos n\theta + x^{-n}$ by u_n, we may write this identity

$$u_n = (x^{n-1} + x^{-n+1})\,u_1 + 2u_{n-1}\cos\theta - u_{n-2};$$

this equation shews that u_n is divisible by u_1, provided u_{n-1} and u_{n-2} are divisible by u_1.

Now $\quad u_2 = (x - 2\cos\theta + x^{-1})(x + 2\cos\theta + x^{-1})$,

hence u_2 is divisible by u_1, and therefore u_3, and so on.

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[1] This method was given by Ferrers in Vol. v. of the *Messenger of Mathematics*.

Hence u_n is divisible by u_1, and therefore $x^2 - 2x\cos\theta + 1$ is a factor of $x^{2n} - 2x^n\cos n\theta + 1$; since θ can be changed into $\theta + \frac{2r\pi}{n}$ without altering $\cos n\theta$ we see that, when r is any integer,

$$x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1$$

is a factor of the given expression; if we let $r = 0, 1, 2 \ldots n-1$ we obtain n different factors of the given expression, and these are all the factors, hence

$$x^{2n} - 2x^n\cos n\theta + 1 = \prod_{r=0}^{r=n-1}\left\{x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1\right\} \ldots (22);$$

this may also be written

$$x^{2n} - 2x^n y^n\cos n\theta + y^{2n} = \prod_{r=0}^{r=n-1}\left\{x^2 - 2xy\cos\left(\theta + \frac{2r\pi}{n}\right) + y^2\right\} \ldots (23).$$

90. In the equation (22) put $\theta = 0$, we have then

$$(x^n - 1)^2 = \prod_{r=0}^{r=n-1}\left(x^2 - 2x\cos\frac{2r\pi}{n} + 1\right),$$

and since $\cos\frac{2r\pi}{n} = \cos\frac{2(n-r)\pi}{n}$, the factors on the right-hand side of this equation are equal in pairs, except that when n is even there is the single factor $x^2 + 2x + 1$, and whether n is even or odd, there is the single factor $x^2 - 2x + 1$, hence

$$x^n - 1 = (x^2 - 1)\prod_{r=1}^{r=\frac{1}{2}(n-2)}\left(x^2 - 2x\cos\frac{2r\pi}{n} + 1\right) \ldots\ldots (24),$$

when n is even, and

$$x^n - 1 = (x - 1)\prod_{r=1}^{r=\frac{1}{2}(n-1)}\left(x^2 - 2x\cos\frac{2r\pi}{n} + 1\right) \ldots\ldots (25),$$

when n is odd.

Again, putting $\theta = \pi/n$ in the formula (22), we have

$$(x^n + 1)^2 = \prod_{r=0}^{r=n-1}\left\{x^2 - 2x\cos\frac{(2r+1)\pi}{n} + 1\right\};$$

now $$\cos\frac{(2r+1)\pi}{n} = \cos\frac{2(n-r)-1}{n}\pi,$$

hence the factors are equal in pairs, except that when n is odd we have the single factor $x^2 + 2x + 1$; hence

$$x^n + 1 = \prod_{r=0}^{r=\frac{1}{2}n-1}\left\{x^2 - 2x\cos\frac{(2r+1)\pi}{n} + 1\right\} \ldots\ldots (26),$$

when n is even, and

$$x^n+1=(x+1)\prod_{r=0}^{r=\frac{1}{2}(n-3)}\left\{x^2-2x\cos\frac{(2r+1)\pi}{n}+1\right\}\ldots(27),$$

when n is odd.

91. In the equation (22) put $x=1$, we have then

$$1-\cos n\theta=2^{n-1}\prod_{r=0}^{r=n-1}\left\{1-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\};$$

changing θ into 2θ this becomes

$$\sin^2 n\theta=2^{2n-2}\sin^2\theta\sin^2\left(\theta+\frac{\pi}{n}\right)\sin^2\left(\theta+\frac{2\pi}{n}\right)\ldots\sin^2\left(\theta+\frac{\overline{n-1}\pi}{n}\right),$$

$$\text{or }\sin n\theta=\pm 2^{n-1}\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{\overline{n-1}\pi}{n}\right),$$

where the ambiguous sign is as yet undetermined. It has been shewn, in Art. 51, that the form of the expansion of $\sin n\theta$ in terms of $\sin\theta$ and $\cos\theta$ is definite; the sign of the product on the right-hand side is therefore always the same; put then $\theta=\pi/2n$, the sign to be taken is clearly positive as each factor is positive. We have therefore

$$\sin n\theta=2^{n-1}\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{\overline{n-1}\pi}{n}\right)\ldots(28).$$

In (28) change θ into $\theta+\pi/2n$, we thus obtain

$$\cos n\theta=2^{n-1}\sin\left(\theta+\frac{\pi}{2n}\right)\sin\left(\theta+\frac{3\pi}{2n}\right)\ldots\sin\left(\theta+\frac{\overline{2n-1}\pi}{2n}\right)\ldots(29).$$

The theorem (18) can be deduced from (28) by putting $\theta=0$, and taking the square root. In a similar manner, the theorem (15) may be deduced from (29).

EXAMPLES.

(1) *Prove that if* n *be an odd integer,* $\sin n\theta+\cos n\theta$ *is divisible by* $\sin\theta+\cos\theta$*, or else by* $\sin\theta-\cos\theta$*.*

Let $$u_n=\sin n\theta+\cos n\theta,$$

then $$u_n+u_{n-4}=2\cos 2\theta\,.\,u_{n-2}=2(\cos^2\theta-\sin^2\theta)\,u_{n-2}.$$

Hence, if u_{n-4} is divisible by $\cos\theta+\sin\theta$ or by $\cos\theta-\sin\theta$, u_n is divisible by the same quantity. Now $u_1=\sin\theta+\cos\theta$, hence u_5, u_9, u_{13} ... are all divisible by $\sin\theta+\cos\theta$; also $u_{-1}=\cos\theta-\sin\theta$, hence u_3, u_7, u_{11} ... are all divisible by $\cos\theta-\sin\theta$.

(2) *Factorise* $\tan n\theta-\tan n\alpha$*.*

We have $$\tan n\theta-\tan n\alpha=\frac{\sin n(\theta-\alpha)}{\cos n\theta\cos n\alpha}.$$

In the formula (28) write $\alpha-\theta$ for θ, we then have

$$\sin n(\theta-\alpha)=(-1)^{n-1}2^{n-1}\prod_{r=0}^{r=n-1}\sin\left(\theta-\alpha-\frac{r\pi}{n}\right)$$

$$=(-1)^{n-1}2^{n-1}\cos^n\theta\prod_{r=0}^{r=n-1}\cos\left(\alpha+\frac{r\pi}{n}\right)\left\{\tan\theta-\tan\left(\alpha+\frac{r\pi}{n}\right)\right\}$$

$$=(-1)^{n-1}\cos^n\theta\sin n\left(\alpha+\frac{\pi}{2}\right)\prod_{r=0}^{r=n-1}\left\{\tan\theta-\tan\left(\alpha+\frac{r\pi}{n}\right)\right\}.$$

Again, we have from (16) and (17)

$$\cos n\theta=\cos\theta\prod_{r=1}^{r=\frac{1}{2}(n-1)}\left(1-\frac{\sin^2\theta}{\sin^2\frac{(2r-1)\pi}{2n}}\right)\text{ or }\prod_{r=1}^{r=\frac{1}{2}n}\left(1-\frac{\sin^2\theta}{\sin^2\frac{(2r-1)\pi}{2n}}\right)$$

according as n is odd or even. Now $1-\dfrac{\sin^2\theta}{\sin^2\beta}=\cos^2\theta\left(1-\dfrac{\tan^2\theta}{\tan^2\beta}\right)$, hence the expression for $\cos n\theta$ may be written

$$\cos^n\theta\prod_{r=1}^{r=\frac{1}{2}(n-1)}\left(1-\frac{\tan^2\theta}{\tan^2\frac{(2r-1)\pi}{2n}}\right)\text{ or }\cos^n\theta\prod_{r=1}^{r=\frac{1}{2}n}\left(1-\frac{\tan^2\theta}{\tan^2\frac{(2r-1)\pi}{2n}}\right).$$

We have therefore

$$\tan n\theta-\tan n\alpha=(-1)^{n-1}\frac{\sin n\left(\alpha+\frac{\pi}{2}\right)}{\cos n\alpha}\,\frac{\displaystyle\prod_{r=0}^{r=n-1}\left\{\tan\theta-\tan\left(\alpha+\frac{r\pi}{n}\right)\right\}}{\displaystyle\prod_{r=1}\left(1-\frac{\tan^2\theta}{\tan^2\frac{(2r-1)\pi}{2n}}\right)},$$

the product in the denominator being taken up to $r=\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

EXAMPLES ON CHAPTER VII.

1. Prove that, if n be an odd positive integer, and $\alpha=\pi/n$,

$$\tan n\phi=(-1)^{\frac{1}{2}(n-1)}\tan\phi\tan(\phi+\alpha)\ldots\ldots\tan(\phi+\overline{n-1}\,\alpha),$$

and $$n\tan n\phi=\tan\phi+\tan(\phi+\alpha)+\ldots\ldots+\tan(\phi+\overline{n-1}\,\alpha).$$

2. Prove that

$$\frac{\sin 5\theta-\cos 5\theta}{\sin 5\theta+\cos 5\theta}=\tan\left(\theta-\tfrac{1}{4}\pi\right)\frac{1-2\sin 2\theta-4\sin^2 2\theta}{1+2\sin 2\theta-4\sin^2 2\theta}.$$

3. Prove that

$$n\cot n\alpha=\cot\alpha+\cot\left(\alpha+\frac{\pi}{n}\right)+\ldots\ldots+\cot\left(\alpha+\frac{\overline{n-1}\,\pi}{n}\right),$$

n being an integer.

4. If $\phi = \pi/13$, shew that

$$\cos\phi + \cos 3\phi + \cos 9\phi = \tfrac{1}{4}(1+\sqrt{13}),$$

and
$$\cos 5\phi + \cos 7\phi + \cos 11\phi = \tfrac{1}{4}(1-\sqrt{13}).$$

5. Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \left(\frac{1}{2}\right)^7.$$

6. Prove that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}.$

Form the cubic of which the roots are

$$\cos\frac{2\pi}{7},\quad \cos\frac{4\pi}{7},\quad \cos\frac{8\pi}{7}.$$

7. Prove that the roots of the equation

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$$

are tan 20°, tan 80°, tan 140°.

8. Prove that

$$\sin^4 a + \sin^4 3a + \sin^4 7a + \sin^4 9a + \sin^4 11a + \sin^4 13a + \sin^4 17a + \sin^4 19a = 3\tfrac{1}{4},$$

where $a = \pi/20$.

9. Prove that

$$2^{n-1}\sin\phi\sin\left(\phi+\frac{2\pi}{n}\right)\sin\left(\phi+\frac{4\pi}{n}\right)\ldots\ldots\sin\left(\phi+\frac{\overline{2n-1}\pi}{n}\right)$$
$$=\cos\frac{n\pi}{2} - \cos n\left(\phi+\frac{\pi}{2}\right).$$

10. Prove that

$$\tan a + \tan\left(\frac{\pi}{2n} - a\right) + \tan\left(\frac{2\pi}{2n} + a\right) + \tan\left(\frac{3\pi}{2n} - a\right) + \ldots\ldots$$

to $2n$ terms is equal to $2n \operatorname{cosec} 2na$.

11. Prove that

$$\sin\frac{2\pi}{2n}\sin\frac{4\pi}{2n}\ldots\ldots\sin\frac{\overline{n-4}\pi}{2n}\sin\frac{\overline{n-2}\pi}{2n}\sin\frac{n\pi}{2n} = \sqrt{\frac{n}{2^{n-1}}},$$

where n is an even positive integer.

12. Prove that

$$\frac{n}{2n-1} = \frac{\sin^2\dfrac{\pi}{2n}}{\sin^2\dfrac{\pi}{2n-1}}\cdot\frac{\sin^2\dfrac{2\pi}{2n}}{\sin^2\dfrac{2\pi}{2n-1}}\ldots\ldots\frac{\sin^2\dfrac{(n-1)\pi}{2n}}{\sin^2\dfrac{(n-1)\pi}{2n-1}},$$

where n is any positive integer.

13. Prove that

$$\frac{\sin n\phi\sin n\theta}{\sin\phi\sin\theta}$$
$$= 2^{n-1}\left\{\cos(\phi-\theta) - \cos\left(\phi+\theta+\frac{2\pi}{n}\right)\right\}\left\{\cos(\phi-\theta) - \cos\left(\phi+\theta+\frac{4\pi}{n}\right)\right\}\ldots\ldots,$$

the number of factors on the right-hand side being $n-1$.

14. Prove that $m \sin n\theta - n \sin m\theta$ is divisible by $\sin^3\theta$, if m and n are odd integers.

15. Shew that if m is a positive integer, $\sec^{2m} A + \operatorname{cosec}^{2m} A$ can be expressed in a series of powers of $\operatorname{cosec} 2A$.

16. Prove that $n = \dfrac{\sin 2a \sin 4a \ldots\ldots \sin (2n-2)a}{\sin a \sin 3a \ldots\ldots \sin (2n-1)a}$,

where $a = \pi/2n$.

17. Prove that

$$(1) \quad \frac{\sin^2 x}{\sin (x-a) \sin (x-b) \sin (x-c)} = \Sigma \frac{\sin^2 a}{\sin (x-a) \sin (a-b) \sin (a-c)},$$

$$(2) \quad \frac{\sin x}{\sin (x-a) \sin (x-b) \sin (x-c)} = \Sigma \frac{\cos (x-a) \sin a}{\sin (x-a) \sin (a-b) \sin (a-c)}.$$

18. Prove that the product of

$$1+\cos a, \quad 1+\cos\left(a+\frac{4\pi}{n}\right) \ldots\ldots 1+\cos\left(a+\frac{4\overline{n-1}\,\pi}{n}\right)$$

is

$$2^{2-n}\{(-1)^{\frac{1}{2}n} - \cos \tfrac{1}{2}na\}^2 \quad \text{or} \quad 2^{1-n}(1+\cos na),$$

according as n is even or odd.

19. Prove that

$$n^2 = \left(\operatorname{versin}\frac{\pi}{2n}\right)^{-1} + \left(\operatorname{versin}\frac{3\pi}{2n}\right)^{-1} + \left(\operatorname{versin}\frac{5\pi}{2n}\right)^{-1} + \ldots\ldots,$$

n terms being taken on the right-hand side.

20. Prove that

$$(\tan 7\tfrac{1}{2}^\circ + \tan 37\tfrac{1}{2}^\circ + \tan 67\tfrac{1}{2}^\circ)(\tan 22\tfrac{1}{2}^\circ + \tan 52\tfrac{1}{2}^\circ + \tan 82\tfrac{1}{2}^\circ) = 17 + 8\sqrt{3}$$

21. Shew that, if m is odd,

$$\tan m\phi = \tan\phi \cot\left(\phi+\frac{\pi}{2m}\right)\tan\left(\phi+\frac{2\pi}{2m}\right)\ldots\ldots$$
$$\ldots\ldots \cot\left(\phi+\frac{\overline{m-2}\,\pi}{2m}\right)\tan\left(\phi+\frac{\overline{m-1}\,\pi}{2m}\right)$$

22. If $28a = \pi$, shew that

$$\sqrt{14} = 2^{13} \sin a \sin 2a \ldots\ldots \sin 13a,$$

and

$$\cos 2a + \cos 6a + \cos 18a - \tfrac{1}{2}\sqrt{7}.$$

23. Prove that $\tan\dfrac{\pi}{2n}\tan\dfrac{2\pi}{2n}\ldots\ldots\tan\dfrac{\overline{n-1}\,\pi}{2n} = 1,$

n being any positive integer.

24. Prove that

$$\operatorname{cosec} x + \operatorname{cosec}\left(x+\frac{2\pi}{n}\right) + \ldots\ldots + \operatorname{cosec}\left(x+\frac{2\overline{n-1}\,\pi}{n}\right)$$
$$= n\{\operatorname{cosec} nx + \operatorname{cosec}(nx+\pi) + \ldots\ldots + \operatorname{cosec}(nx+\overline{n-1}\,\pi)\}.$$

25. Prove that, according as n is even or odd,

$$2(1+\cos n\theta) \text{ or } (1+\cos n\theta)/(1+\cos\theta)$$

is the square of a rational integral function of $2\cos\theta$. Shew that

$$1+\cos 9\theta=(1+\cos\theta)(16\cos^4\theta-8\cos^3\theta-12\cos^2\theta+4\cos\theta+1)^2.$$

26. Prove that $2^{n-1}\cos^n\theta-\cos n\theta$ is divisible by $1+2\cos 2\theta$, when n is of the form $6m-1$, and by $(1+2\cos 2\theta)^2$, when n is of the form $6m+1$, m being a positive integer.

Prove that

$$2^{10}\cos^{11}\theta-\cos 11\theta=11\cos\theta\,(1+2\cos 2\theta)\,\{(1+2\cos 2\theta)^3+(1+2\cos 2\theta)+1\}.$$

27. Prove that, if n be an odd positive integer, and

$$\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)=\tan^n(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta),$$

then
$$\sin\phi=n\sin\theta\prod_{r=1}^{r=\frac{1}{2}(n-1)}\left\{\frac{1+\sin^2\theta\cot^2\dfrac{r\pi}{n}}{1+\sin^2\theta\tan^2\dfrac{r\pi}{n}}\right\}.$$

28. Shew that any function of the form $f(\sin\theta,\cos\theta)/\phi(\sin\theta,\cos\theta)$, where f and ϕ denote rational integral functions of degree n, containing $\cos^n\theta$, can be expressed in the form $A\Pi\sin\frac{1}{2}(\theta-a)/\Pi\sin\frac{1}{2}(\theta-a')$, where A and the quantities a, a' are independent of θ, and there are $2n$ factors in the numerator and $2n$ in the denominator.

If the function $\dfrac{a\cos 2\theta+b\cos\theta+c\sin\theta+d}{a'\cos 2\theta+b'\cos\theta+c'\sin\theta+d'}$ be expressed in this form, prove that Σa and $\Sigma a'$ are even multiples of π.

29. Prove that

$$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}.$$

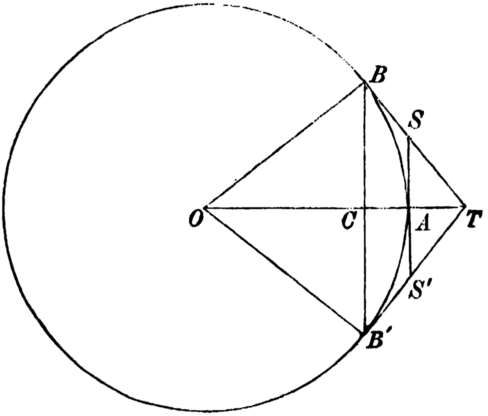
30. Prove that

$$\frac{2^6\sin^7\theta+\sin 7\theta}{2^6\cos^7\theta-\cos 7\theta}=\tan\theta\tan^2\left(\theta+\frac{\pi}{6}\right)\tan^2\left(\theta-\frac{\pi}{6}\right).$$

CHAPTER VIII.

RELATIONS BETWEEN THE CIRCULAR FUNCTIONS AND THE CIRCULAR MEASURE OF AN ANGLE.

92. WE shall now investigate theorems which assign certain limits between which the sine, cosine and tangent of an angle whose circular measure θ is less than $\frac{1}{2}\pi$ must lie. The first theorem which we shall prove is that *if* θ *be the circular measure of an angle less than* $\frac{1}{2}\pi$, *then* $\sin\theta < \theta < \tan\theta$, *unless* $\theta = 0$.



Let $AOB = AOB' = \theta$; and let TB, TB' be the tangents at B and B', and let SAS' be the tangent at A. In Art. 11, it was shewn that the length of the arc AB does not exceed $AS+SB$; and thus the arc BAB' does not exceed $BS+B'S'+SS'$, and therefore arc $BAB' < BT + TB'$; or arc $BA < BT$. Also

$$\text{arc } BA > BA > BC.$$

Consequently we have

$$BC/OB < \text{arc } BA/OB < BT/OB.$$

Now $\theta = \text{arc } BA/OB$, $\sin\theta = BC/OB$, and $\tan\theta = BT/OB$;

therefore $\sin\theta < \theta < \tan\theta$. If θ had been greater than $\frac{1}{2}\pi$, T might have been on the other side of O, and the inequalities which we have employed would not necessarily hold.

Since $\sin\theta < \theta < \tan\theta$, we have $1 < \theta/\sin\theta < \sec\theta$; now suppose θ to be indefinitely diminished, then the limit when $\theta = 0$ of $\sec\theta$ is 1; hence also the limit of $\theta/\sin\theta$, when θ is indefinitely diminished, is unity. Since

$$\frac{\sin\theta}{\theta} = (\theta \operatorname{cosec}\theta)^{-1}, \quad \text{and} \quad \frac{\tan\theta}{\theta} = \sec\theta \,.\, (\theta \operatorname{cosec}\theta)^{-1},$$

we have the theorems that *the limits of* $\frac{\sin\theta}{\theta}$ *and* $\frac{\tan\theta}{\theta}$, *when* θ *is indefinitely diminished, are each unity.*

The theorem may also be proved thus:—The triangle OAB, the sector OAB, and the triangle OBT are in ascending order of magnitude; and $\Delta OAB = \frac{1}{2}OA \,.\, BC = \frac{1}{2}OA^2 \sin\theta$, also sector $OAB = \frac{1}{2}OA^2 \,.\, \theta$, and

$$\Delta OBT = \tfrac{1}{2}OB \,.\, BT = \tfrac{1}{2}OB^2 \,.\, \tan\theta,$$

therefore $\sin\theta < \theta < \tan\theta$.

93. The reason, to which we referred in Art. 5, why the circular measure is more convenient in Analytical Trigonometry than any other measure of an angle, is that in this measure the sine and tangent of an angle are each ultimately equal to the angle itself, as the angle is diminished indefinitely; whereas if we use any other measure, as for instance seconds, this is not the case. We have in the case of seconds

$$\frac{\sin n''}{n''} = \frac{\sin\theta}{\theta} \times \frac{\pi}{180 \times 60 \times 60},$$

$$\frac{\tan n''}{n''} = \frac{\tan\theta}{\theta} \times \frac{\pi}{180 \times 60 \times 60},$$

where θ is the circular measure of n seconds, hence the limits of $\frac{\sin n''}{n''}$, $\frac{\tan n''}{n''}$ when n is indefinitely diminished are each equal to $\frac{\pi}{180 \times 60 \times 60}$. If then we used seconds instead of circular measure, we should constantly have the number $\frac{\pi}{180 \times 60 \times 60}$ occurring, instead of unity, in the large class of formulae which involve the limits of $\frac{\sin\theta}{\theta}$ and $\frac{\tan\theta}{\theta}$ for $\theta = 0$.

The limits of $m \sin \frac{a}{m}$, $m \tan \frac{a}{m}$ are each a, when m is indefinitely increased, for $m \sin \frac{a}{m} = a\left(\frac{\sin\theta}{\theta}\right)$, $m \tan \frac{a}{m} = a\left(\frac{\tan\theta}{\theta}\right)$, where $\theta = \frac{a}{m}$, and when m is indefinitely increased, θ becomes indefinitely small. The limiting values of $\frac{\sin p\theta}{\sin q\theta}$, $\frac{\tan p\theta}{\tan q\theta}$, when θ is indefinitely diminished, are each equal to p/q.

94. Since, if $\theta < \frac{1}{2}\pi$, $\tan \frac{1}{2}\theta > \frac{1}{2}\theta$, we have $\sin \frac{1}{2}\theta > \frac{1}{2}\theta \cos \frac{1}{2}\theta$,

hence $$2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta > \theta \cos^2 \tfrac{1}{2}\theta,$$

or $$\sin\theta > \theta\,(1 - \sin^2 \tfrac{1}{2}\theta).$$

Now $$\sin^2 \tfrac{1}{2}\theta < (\tfrac{1}{2}\theta)^2,$$

hence $$\sin\theta > \theta\,(1 - \tfrac{1}{4}\theta^2), \quad \text{or} \quad \sin\theta > \theta - \tfrac{1}{4}\theta^3.$$

Also $\cos\theta = 1 - 2\sin^2 \frac{1}{2}\theta$, and this is greater than $1 - 2\,(\frac{1}{2}\theta)^2$; or $\cos\theta > 1 - \frac{1}{2}\theta^2$. Also, since $\sin \frac{1}{2}\theta > \frac{1}{2}\theta - \frac{1}{4}\,(\frac{1}{2}\theta)^3$, we have

$$\cos\theta < 1 - 2\,(\tfrac{1}{2}\theta - \tfrac{1}{32}\theta^3)^2 < 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{16}\theta^4 - 2\,\frac{\theta^6}{32^2},$$

hence $\cos\theta < 1 - \frac{1}{2}\theta^2 + \frac{1}{16}\theta^4$. We may state the results we have obtained thus:—

If θ be the circular measure of an angle less than $\frac{1}{2}\pi$, then $\sin\theta$ *lies between θ and $\theta - \frac{1}{4}\theta^3$, and* $\cos\theta$ *lies between*

$$1 - \tfrac{1}{2}\theta^2 \quad \textit{and} \quad 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{16}\theta^4.$$

95. We shall now shew that if $\theta < \frac{1}{2}\pi$,

$$\sin\theta > \theta - \tfrac{1}{6}\theta^3, \quad \cos\theta < 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4.$$

This makes the limits for $\sin\theta$ and $\cos\theta$ closer than in the theorems of the last article.

We have $$3 \sin \tfrac{1}{3}\theta - \sin\theta = 4 \sin^3 \tfrac{1}{3}\theta,$$

$$3 \sin \frac{\theta}{3^2} - \sin \frac{\theta}{3} = 4 \sin^3 \frac{\theta}{3^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$3 \sin \frac{\theta}{3^n} - \sin \frac{\theta}{3^{n-1}} = 4 \sin^3 \frac{\theta}{3^n}.$$

Multiply these equations by 1, 3, 3^2 ...... 3^{n-1} respectively, and then add them, we have

$$3^n \sin \frac{\theta}{3^n} - \sin\theta = 4\left(\sin^3 \frac{\theta}{3} + 3 \sin^3 \frac{\theta}{3^2} + \ldots + 3^{n-1} \sin^3 \frac{\theta}{3^n}\right),$$

hence $$\theta\cdot\frac{\sin\frac{\theta}{3^n}}{\frac{\theta}{3^n}}-\sin\theta<4\left(\frac{\theta^3}{3^3}+\frac{\theta^3}{3^5}+\dots+\frac{\theta^3}{3^{2n+2}}\right)$$

$$<\frac{4}{3^3}\theta^3\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\dots+\frac{1}{3^{2n-2}}\right).$$

Now let n be increased indefinitely, then the limit of $\dfrac{\sin\frac{\theta}{3^n}}{\frac{\theta}{3^n}}$ is unity, and of the series $1+\dfrac{1}{3^2}+\dfrac{1}{3^4}+\dots$ is $\dfrac{1}{1-\frac{1}{3^2}}=\dfrac{9}{8}$; therefore

$$\theta-\sin\theta<\tfrac{1}{6}\theta^3,\quad\text{or}\quad\sin\theta>\theta-\tfrac{1}{6}\theta^3.$$

Also $$\cos\theta=1-2\sin^2\tfrac{1}{2}\theta;$$

therefore $$\cos\theta<1-2\left(\tfrac{1}{2}\theta-\tfrac{1}{48}\theta^3\right)^2<1-\tfrac{1}{2}\theta^2+\tfrac{1}{24}\theta^4.$$

Hence $\sin\theta$ *lies between* θ *and* $\theta-\frac{1}{6}\theta^3$, *and* $\cos\theta$ *lies between* $1-\frac{1}{2}\theta^2$ *and* $1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4$, *the angle* θ *being less than* $\frac{1}{2}\pi$.

We have also $\tan\theta=\sin\theta/\cos\theta$, hence

$$\tan\theta>(\theta-\tfrac{1}{6}\theta^3)(1-\tfrac{1}{2}\theta^2)^{-1}>(\theta-\tfrac{1}{6}\theta^3)(1+\tfrac{1}{2}\theta^2+\tfrac{1}{4}\theta^4),$$

or $\tan\theta>\theta+\frac{1}{3}\theta^3+\frac{1}{2}\theta^5-\frac{1}{24}\theta^7$, therefore $\tan\theta>\theta+\frac{1}{3}\theta^3$.

Euler's product.

96. We have $$\sin\theta=2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta,$$

$$\sin\frac{\theta}{2}=2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2},$$

$$\sin\frac{\theta}{2^2}=2\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3},$$

....

$$\sin\frac{\theta}{2^{n-1}}=2\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n},$$

hence $$\sin\theta=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\dots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}.$$

Now when n is indefinitely increased, the limit of $2^n \sin \frac{\theta}{2^n}$ is θ: hence *the limit, when* n *is indefinitely increased, of the product*

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\dots\cos\frac{\theta}{2^n}, \textit{ is } \frac{\sin\theta}{\theta}.$$

In this product put $\theta=\frac{1}{2}\pi$, we then obtain Vieta's expression for π, viz

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\dots\dots$$

EXAMPLES.

(1) *Prove that as θ increases from 0 to $\frac{1}{2}\pi$, $\frac{\sin\theta}{\theta}$ continually diminishes, and $\frac{\tan\theta}{\theta}$ continually increases.*

We shall shew that $\frac{\sin\theta}{\theta}>\frac{\sin(\theta+h)}{\theta+h}$; that is

$$(\theta+h)\sin\theta>\theta(\sin\theta\cos h+\cos\theta\sin h), \text{ or } \frac{\tan\theta}{\theta}>\frac{\sin h}{h+(1-\cos h)\theta}.$$

Now we know that $\frac{\tan\theta}{\theta}>1>\frac{\sin h}{h}$, and $\frac{\sin h}{h}>\frac{\sin h}{h+(1-\cos h)\theta}$, since $1-\cos h$ is positive, hence the inequality is established; thus $\frac{\sin\theta}{\theta}$ diminishes from 1 to $2/\pi$, as θ increases from 0 to $\frac{1}{2}\pi$.

We shall next shew that

$$\frac{\tan(\theta+h)}{\theta+h}>\frac{\tan\theta}{\theta}, \text{ or } \theta\sin(\theta+h)\cos\theta>(\theta+h)\sin\theta\cos(\theta+h);$$

this is equivalent to

$$\theta\sin h>h\sin\theta\cos(\theta+h), \text{ or } \frac{\sin h}{h}>\frac{\sin\theta}{\theta}\cos(\theta+h);$$

now we may suppose $h<\theta$, hence by the first theorem

$$\frac{\sin h}{h}>\frac{\sin\theta}{\theta}, \text{ and therefore } \frac{\sin\theta}{\theta}>\frac{\sin\theta}{\theta}\cos(\theta+h).$$

Thus as θ increases from 0 to $\frac{1}{2}\pi$, $\frac{\tan\theta}{\theta}$ increases from 1 to ∞. The theorems may be seen to be true by referring to the graphs of $\sin\theta$, $\cos\theta$ given in Art. 32; it will be seen that in the first case the ratio of the ordinate to the abscissa diminishes, and in the second case increases, as θ increases from 0 to $\frac{1}{2}\pi$.

(2) *Prove that the equation $\tan x=\lambda x$ has an infinite number of real roots, and find the approximate values of the large roots.*

In Art. 32 we have drawn the graph of the function $\tan x$; draw in the same figure the graph of λx, this is a straight line through the point O. The straight line will obviously intersect each branch of the graph of $\tan x$, and the values of x corresponding to these points of intersection are the solutions of the equation. There is therefore a root of the equation between

$$x=(2k-1)\frac{\pi}{2} \text{ and } (2k+1)\frac{\pi}{2},$$

where k is any integer. If $k\lambda$ be large, then $(2k+1)\frac{\pi}{2}$ is obviously an approximate solution; to find a nearer approximation let $x=(2k+1)\frac{\pi}{2}+y$, where y is small, then $-\cot y=\lambda y+(2k+1)\frac{\lambda\pi}{2}$, putting $\cos y=1$, $\sin y=y$, and neglecting y^2, we have

$$-1=(2k+1)\frac{\lambda\pi}{2}y, \text{ or } y=-\frac{2}{(2k+1)\lambda\pi}, \text{ therefore } x=(2k+1)\frac{\pi}{2}-\frac{2}{(2k+1)\lambda\pi}$$

is the approximate solution. To find a still nearer approximation, neglect y^3, putting $y=-\frac{2}{(2k+1)\lambda\pi}$ in the terms which involve y^2, we have

$$\tfrac{1}{2}y^2-1=\left\{\lambda y+(2k+1)\frac{\lambda\pi}{2}\right\}y=\lambda y^2+y(2k+1)\frac{\lambda\pi}{2},$$

hence
$$y(2k+1)\frac{\lambda\pi}{2}=-1+(\tfrac{1}{2}-\lambda)\frac{4}{(2k+1)^2\lambda^2\pi^2},$$

or $y=-\frac{2}{(2k+1)\lambda\pi}+(\frac{1}{2}-\lambda)\frac{8}{(2k+1)^3\lambda^3\pi^3}$; the approximate value of x is therefore $x=(2k+1)\frac{\pi}{2}-\frac{2}{(2k+1)\lambda\pi}+(\frac{1}{2}-\lambda)\frac{8}{(2k+1)^3\lambda^3\pi^3}$.

(3) *Prove that* $\frac{1}{\theta}=\cot\theta+\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{4}\tan\frac{\theta}{4}+\frac{1}{8}\tan\frac{\theta}{8}+\ldots$ *ad inf.*

It can easily be shewn that

$$\tfrac{1}{2}\cot\frac{\theta}{2}-\cot\theta=\tfrac{1}{2}\tan\frac{\theta}{2},$$

hence also
$$\tfrac{1}{4}\cot\frac{\theta}{4}-\tfrac{1}{2}\cot\frac{\theta}{2}=\tfrac{1}{4}\tan\frac{\theta}{4},$$

……………………………………………

$$\frac{1}{2^{2n}}\cot\frac{\theta}{2^{2n}}-\frac{1}{2^{2n-1}}\cot\frac{\theta}{2^{2n-1}}=\frac{1}{2^{2n}}\tan\frac{\theta}{2^{2n}},$$

hence by addition we have

$$\tfrac{1}{2}\tan\frac{\theta}{2}+\frac{1}{2^2}\tan\frac{\theta}{2^2}+\ldots+\frac{1}{2^{2n}}\tan\frac{\theta}{2^{2n}}=\frac{1}{2^{2n}}\cot\frac{\theta}{2^{2n}}-\cot\theta.$$

Now when n is indefinitely increased, the limiting value of $\frac{1}{2^{2n}}\cot\frac{\theta}{2^{2n}}$ is $\frac{1}{\theta}$, hence the limiting sum of the series is $\frac{1}{\theta}-\cot\theta$.

If we put $\theta=\frac{1}{2}\pi$, we obtain the theorem

$$\frac{1}{\pi}=\tfrac{1}{4}\tan\frac{\pi}{4}+\tfrac{1}{8}\tan\frac{\pi}{8}+\tfrac{1}{16}\tan\frac{\pi}{16}+\ldots.$$

The limits of certain expressions.

97. When n is indefinitely increased, the limits of each of the expressions $\cos\frac{\theta}{n}$, $\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}$ is unity; hence the limiting values of $\left(\cos\frac{\theta}{n}\right)^r$, $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^r$ are also unity *provided* r *is any number which is independent of* n; but if r is a function $f(n)$ of n, which becomes infinite when n does so, the expressions $\left(\cos\frac{\theta}{n}\right)^{f(n)}$, $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^{f(n)}$ are undetermined forms of the class 1^∞, and the values of their limits depend upon the form of $f(n)$.

To determine the limiting values of $\left(\cos\frac{\theta}{n}\right)^{f(n)}$, we have, denoting the expression by u,

$$\log_e u = \tfrac{1}{2}f(n)\log_e\left(1-\sin^2\frac{\theta}{n}\right).$$

It will be assumed as a known theorem that the limit, when x becomes indefinitely small, of $\dfrac{\log_e(1-x)}{x}$ is -1. Then, since

$$\log_e u = \tfrac{1}{2}f(n)\sin^2\frac{\theta}{n}\cdot\frac{\log_e\left(1-\sin^2\frac{\theta}{n}\right)}{\sin^2\frac{\theta}{n}}$$

the limit of $\log_e u$ is equal to that of $\frac{1}{2}f(n)\sin^2\frac{\theta}{n}$, with its sign changed, provided this latter limit exists. We can find the limit of $\log_e u$, and therefore of u, in the following cases:—

(1) If $f(n)=n$; then $f(n)\sin^2\frac{\theta}{n}=n\sin\frac{\theta}{n}\cdot\sin\frac{\theta}{n}$, and the limit of $n\sin\frac{\theta}{n}$ is θ, and that of $\sin\frac{\theta}{n}$ is zero; therefore the limit of $\log_e u$ is zero, or that of u is 1.

(2) If $f(n)=n^2$; then $f(n)\sin^2\frac{\theta}{n}=\left(n\sin\frac{\theta}{n}\right)^2$, of which the limit is θ^2. Hence the limit of $\log_e u$ is $-\frac{1}{2}\theta^2$, or that of u is $e^{-\frac{1}{2}\theta^2}$.

(3) $f(n)=n^p$, where $p>2$, then $f(n)\sin^2\frac{\theta}{n}=n^{p-2}\cdot\left(n\sin\frac{\theta}{n}\right)^2$, and this increases indefinitely as n does so. Therefore the limit of $\log_e u$ is $-\infty$, hence the limit of u is zero.

98. To find the limiting value of $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^n$; since $\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}$ is less than 1 and greater than $\dfrac{\sin\frac{\theta}{n}}{\tan\frac{\theta}{n}}$ or $\cos\frac{\theta}{n}$, the limit of $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^n$ lies between 1^n or 1, and $\left(\cos\frac{\theta}{n}\right)^n$; thus from case (1) in the last Article, the limit of the expression is unity. We see also that the limiting values of $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^{n^2}$ and of $\left(\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^{n^p}$ $(p>2)$ lie between 1 and $e^{-\frac{1}{6}\theta^2}$, and between 1 and 0, respectively.

Series for the sine and cosine of an angle in powers of its circular measure.

99. In the formulae (39), (40) of Chapter IV. write θ for A, and let $x=n\theta$, we have then

$$\sin x = n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta+\dots$$
$$+(-1)^r\frac{n(n-1)\dots(n-2r)}{(2r+1)!}\cos^{n-2r-1}\theta\sin^{2r+1}\theta+\dots,$$

$$\cos x=\cos^n\theta-\frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta+\dots$$
$$+(-1)^s\frac{n(n-1)\dots(n-2s+1)}{(2s)!}\cos^{n-2s}\theta\sin^{2s}\theta+\dots.$$

We may write these series in the forms

$$\sin x = x\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{x(x-\theta)(x-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \dots$$
$$+(-1)^r\frac{x(x-\theta)\dots(x-2r\theta)}{(2r+1)!}\cos^{n-2r-1}\theta\left(\frac{\sin\theta}{\theta}\right)^{2r+1} + \dots,$$

$$\cos x = \cos^n\theta - \frac{x(x-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \dots$$
$$+(-1)^s\frac{x(x-\theta)\dots(x-\overline{2s-1}\,\theta)}{(2s)!}\cos^{n-2s}\theta\left(\frac{\sin\theta}{\theta}\right)^{2s} + \dots.$$

The number of terms in each of these series depends upon the value of n, and increases indefinitely as n is indefinitely increased. In order to obtain the limits of the expressions when n is indefinitely increased, it is necessary to replace each of these series by a series in which the number of terms is fixed, and does not increase indefinitely with n.

The ratio of one term of the series for $\sin x$ to the immediately preceding term is

$$-\frac{(x-\overline{2r+1}\,\theta)(x-\overline{2r+2}\,\theta)}{(2r+2)(2r+3)}\left(\frac{\tan\theta}{\theta}\right)^2;$$

this number is negative, and is numerically less than

$$\left\{\frac{x^2}{(2r+2)(2r+3)} + \left(\frac{x}{n}\right)^2 + \frac{x^2}{n}\cdot\frac{1}{r+1}\right\}\left(\frac{\tan\theta}{\theta}\right)^2.$$

If x have any fixed value, $\left(\frac{\tan\theta}{\theta}\right)^2$ diminishes as n is increased; values n_1, r_1 of n and r may be so chosen that the above expression has values which are less than unity for $n \geqq n_1$, $r \geqq r_1$. For the fixed value of x, and for all values of n which are $\geqq n_1$, the series for $\sin x$ is such that, from and after a fixed term, the position of which is independent of n, each term is numerically less than the one that precedes it. Since the sum of a series of terms with alternate signs, when each term is numerically less than the preceding one, is less than the first term, we have

$$\sin x = x\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{x(x-\theta)(x-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \dots$$
$$+(-1)^r\epsilon\frac{x(x-\theta)\dots(x-2r\theta)}{(2r+1)!}\cos^{n-2r-1}\theta\left(\frac{\sin\theta}{\theta}\right)^{2r+1},$$

where $\theta = x/n$, provided $n \geqq n_1$; r is independent of n, and ϵ is a number between 0 and 1. The integer r may have any value not less than r_1.

In a similar manner, we can prove that

$$\cos x = \cos^n \theta - \frac{x(x-\theta)}{2!} \cos^{n-2} \theta \left(\frac{\sin\theta}{\theta}\right)^2 + \frac{x(x-\theta)(x-2\theta)(x-3\theta)}{4!} \cos^{n-4} \theta \left(\frac{\sin\theta}{\theta}\right)^4 + \dots + (-1)^s \epsilon' \frac{x(x-\theta)\dots(x-\overline{2s-1}\theta)}{(2s)!} \cos^{n-2s} \theta \left(\frac{\sin\theta}{\theta}\right)^{2s},$$

provided $n \geqq n_1'$; s is independent of n, and ϵ' is a number between 0 and 1.

Now let n be indefinitely increased; the limits of the expressions for $\sin x$, $\cos x$ must represent these functions. Since the number of terms in each of the series is fixed, being independent of n, we have only to add the limits of the several terms in order to obtain the limit of the sum. The limit of $\left(\frac{\sin\theta}{\theta}\right)^k$, where k is independent of n, is unity. Also the limit of $\cos^{n-k}\theta$ is that of $\cos^n\theta$, divided by that of $\cos^k\theta$; and it has been shewn in Art. 97 that $L \cos^n \theta = 1$; that $L \cos^k \theta = 1$ follows from $L \cos\theta = 1$; hence $L \cos^{n-k}\theta = 1$. The numbers ϵ, ϵ' depend upon n, but they are for each value of n between 0 and 1, and therefore their limits $\bar{\epsilon}$, $\bar{\epsilon}'$ cannot exceed unity. We thus have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \bar{\epsilon} \frac{x^{2r+1}}{(2r+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^s \bar{\epsilon}' \frac{x^{2s}}{(2s)!},$$

where $\bar{\epsilon}$, $\bar{\epsilon}'$ are positive numbers which cannot exceed unity.

These results hold, for each value of x, for all values of r and s which are greater than or equal to fixed integers r_1 and s_1. It follows that for each value of x, $\sin x$ is represented by the convergent series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \dots;$$

and $\cos x$ is represented by the convergent series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^m \frac{x^{2m}}{(2m)!} + \dots.$$

For the sum of a fixed number of terms of the first series differs from $\sin x$ by not more than $\dfrac{x^{2r+1}}{(2r+1)!}$, which for each value of x is arbitrarily small if r be chosen sufficiently large. That this is the case is seen by observing that the ratio $\dfrac{x^2}{2r(2r+1)}$ of $\dfrac{x^{2r+1}}{(2r+1)!}$ to $\dfrac{x^{2r-1}}{(2r-1)!}$ may be made arbitrarily small, for any fixed value of x, by taking r great enough. Similar reasoning applies to the expression for $\cos x$.

EXAMPLES.

(1) *Expand* $\cos^3 x$ *in powers of* x

We have $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$; expanding $\cos 3x$, $\cos x$ in powers of x, we find for the general term in the expansion of $\cos^3 x$, $(-1)^n \dfrac{3^{2n}+3}{4(2n)!} x^{2n}$ It will be seen that any integral power of $\cos x$ or $\sin x$, or the product of two such powers, may be expanded in powers of x by putting the expression into the sum of cosines or sines of multiples of x.

(2) *Expand* $\tan x$ *in powers of* x *as far as the term in* x^7.

We have $\tan x = \left\{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040}\right\}\left\{1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720}\right\}^{-1}$, leaving out terms of higher order than x^7. Expanding the second factor, we have

$$\tan x = \left\{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}\right\}\left[1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}\right) + \left(\frac{x^2}{2} - \frac{x^4}{24}\right)^2 + \left(\frac{x^2}{2}\right)^3\right];$$

multiplying out and collecting the coefficients of the terms up to x^7, we find

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7.$$

(3) *Find the limit of* $\dfrac{\sin(\tan x) - \tan(\sin x)}{x^7}$, *when* $x = 0$.

The numerator of the expression is equal to

$\tan x - \frac{1}{6}\tan^3 x + \frac{1}{120}\tan^5 x - \frac{1}{5040}\tan^7 x - \sin x - \frac{1}{3}\sin^3 x - \frac{2}{15}\sin^5 x - \frac{17}{315}\sin^7 x$, using the expansion obtained in the last example. This is equal to

$$(x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7) - \tfrac{1}{6}x^3(1 + x^2 + \tfrac{11}{15}x^4) + \frac{x^5}{120}(1 + \tfrac{5}{3}x^2) - \frac{x^7}{5040}$$
$$-\left(x - \tfrac{1}{6}x^3 + \frac{x^5}{120} - \frac{x^7}{5040}\right) - \frac{x^3}{3}\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^4}{12}\right) - \tfrac{2}{15}x^5(1 - \tfrac{5}{6}x^2) - \tfrac{17}{315}x^7,$$

rejecting all terms of higher order than x^7; this expression reduces to $-\frac{1}{30}x^7$. The limit of the given expression is therefore $-1/30$.

A relation between trigonometrical and algebraical identities.

100. From any trigonometrical identity in which the angles are homogeneous functions of the letters, a series of algebraical identities may be deduced, by expanding the circular functions in powers of the circular measure of the angles, and equating the terms of each order. Thus for example, in the formula $\sin a \sin b = \frac{1}{2}\{\cos(a-b) - \cos(a+b)\}$, expand each of the sines and cosines and equate the terms of the second order, we have then $ab = \frac{1}{4}\{(a+b)^2 - (a-b)^2\}$. In Articles 44 and 47 of Chapter IV., we have given a number of examples of analogous trigonometrical and algebraical identities; in each case the algebraical identity is obtained, as we have above explained, from the trigonometrical one. For example, in example (11), Art. 47, which may be written

$$\Sigma \sin^2 a \sin(b+c-a) - 2\sin a \sin b \sin c$$
$$= \sin(b+c-a)\sin(c+a-b)\sin(a+b-c),$$

if we equate the terms of the third order, when the sines are expanded, we obtain the analogous algebraical identity

$$\Sigma a^2(b+c-a) - 2abc = (b+c-a)(c+a-b)(a+b-c).$$

EXAMPLES ON CHAPTER VIII.

1. Prove geometrically that

$$\tan\theta \geqq 2\tan\tfrac{1}{2}\theta, \quad \text{where} \quad \theta < \tfrac{1}{2}\pi.$$

2. Trace the changes in the value of $\tan 3\theta \cot^3\theta$, as θ increases from 0 to $\frac{1}{2}\pi$

Shew that $17+12\sqrt{2}$ is a minimum and $17-12\sqrt{2}$ a maximum value of the expression.

3. Prove that $\tan 3\theta \cot\theta$ cannot lie between 3 and $1/3$.

4. Prove that $\theta > \dfrac{3\sin\theta}{2+\cos\theta}$, where $\theta < \frac{1}{2}\pi$.

5. Prove that $3\tan 5\theta > 5\tan 3\theta$, if θ lies between 0 and $\pi/10$.

6. Shew that the limiting value of $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\theta^2}$, when $\theta = 0$, is $\frac{1}{3}$.

7. Prove that $\sin(\cos\theta) < \cos(\sin\theta)$ for all values of θ.

8. Prove that the limiting value of the infinite product

$$\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)\ldots\ldots \text{ is } \frac{\theta}{\tan\theta}.$$

9. If $\dfrac{\sin(\theta-\phi)}{\sin\phi} = 1+n$, and n be very small, prove that

$$\sin\phi = (1-\tfrac{1}{2}n)\sin\tfrac{1}{2}\theta, \text{ approximately.}$$

10. Find the limiting value of $\dfrac{\sin(\theta\cos\theta)}{\cos(\theta\sin\theta)}$, when $\theta = \frac{1}{2}\pi$.

11. Find the limiting value, when $\theta = 0$, of $\dfrac{\tan 2\theta - 2\tan\theta}{\theta^3}$

12. Prove that the limiting value of

$$\left(\frac{\cot\theta}{\sqrt{2-2\sin\theta}}\right)^{\tan^2(\frac{1}{4}\pi+\frac{1}{2}\theta)}, \text{ when } \theta = \tfrac{1}{2}\pi, \text{ is } e^{\frac{3}{2}}$$

13. Prove that

$$\left(\frac{\sin x}{x}\right)^2 = 1-\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\sin^2\frac{x}{4}-\cos^2\frac{x}{2}\cos^2\frac{x}{4}\sin^2\frac{x}{8}-\ldots\ldots.$$

14. If in the equation $\tan\theta = \dfrac{1}{\cot\alpha_1+\cot\alpha_2}+\dfrac{1}{\cot\alpha_3+\cot\alpha_4}$, the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be all nearly equal; shew that θ is very nearly equal to

$$\tfrac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4).$$

15. Sum the series

$$\cos\frac{\theta}{2}+2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}+2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}+\ldots\ldots \text{ to } n \text{ terms.}$$

16. Prove that the sum to infinity of the series

$$\tan\frac{x}{2}\sec x+\tan\frac{x}{2^2}\sec\frac{x}{2}+\tan\frac{x}{2^3}\sec\frac{x}{2^2}+\ldots\ldots \text{ is } \tan x.$$

17. Shew that

$$\theta-\sin\theta\cos\theta = 2\sin\theta\sin^2\frac{\theta}{2}+2^2\sin\frac{\theta}{2}\sin^2\frac{\theta}{4}+2^3\sin\frac{\theta}{4}\sin^2\frac{\theta}{8}+\ldots\ldots \textit{ ad inf}$$

18. Prove that $\tan\theta = \cfrac{2}{\cot\frac{\theta}{2}-\cfrac{2}{\cot\frac{\theta}{4}-\cfrac{2}{\cot\frac{\theta}{8}-}}}\ldots\ldots$

19. If $\theta < \pi$, shew that

$$2\left[\sin\frac{\theta}{2}+\sin\frac{\theta}{2^2}+\ldots\ldots+\sin\frac{\theta}{2^n}\right]\left[\cos\frac{\theta}{2}+\cos\frac{\theta}{2^2}+\ldots\ldots+\cos\frac{\theta}{2^n}\right]$$

$$\nless \left[\sin\theta\sin\frac{\theta}{2}\ldots\ldots\sin\frac{\theta}{2^{n-1}}\right]^{\frac{1}{n}}$$

20. If a and b be positive quantities, and if $a_1=\frac{1}{2}(a+b)$, $b_1=(a_1b)^{\frac{1}{2}}$, $a_2=\frac{1}{2}(a_1+b_1)$, $b_2=(a_2b_1)^{\frac{1}{2}}$, and so on, shew that $a_\infty=b_\infty=\dfrac{(b^2-a^2)^{\frac{1}{2}}}{\cos^{-1}\dfrac{a}{b}}$.

Shew that the value of π may be calculated by means of this formula

21. Find the limiting value of the infinite product

$$(\sin\theta\cos\tfrac{1}{2}\theta)^{\frac{1}{2}}(\sin\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta)^{\frac{1}{4}}(\sin\tfrac{1}{4}\theta\cos\tfrac{1}{8}\theta)^{\frac{1}{8}}\ldots\ldots$$

22. If $\tan\theta=4\theta$, the value of θ between 0 and $\frac{1}{2}\pi$ will be

$$\frac{\pi}{2}-\left(\frac{1}{2\pi}+\frac{11}{24\pi^3}+\frac{403}{480\pi^5}+\ldots\ldots\right).$$

23. Prove that $\dfrac{\sin\theta}{1+2\cos\theta}=\displaystyle\sum^{\infty}\left\{\frac{1}{2^n}\,\frac{\sin\dfrac{\theta}{2^n}}{2\cos\dfrac{\theta}{2^n}-1}\right\}$.

24. Prove that

$$\frac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)\ldots\ldots(2\cos 2^{n-1}\theta-1).$$

25. Sum to n terms the series

$$\frac{1}{2}\log\tan 2\theta+\frac{1}{2^2}\log\tan 2^2\theta+\frac{1}{2^3}\log\tan 2^3\theta+\ldots\ldots$$

26. Having given that the limiting value, when $\theta=0$, of $\dfrac{\theta^n\sin^n\theta}{\theta^n-\sin^n\theta}$ is neither zero nor infinite, find n.

27. Find the limit, when $x=0$, of

$$\frac{1-\cos 2x+\cos 4x-\cos 6x+\cos 8x-\cos 10x-\cos 14x+\cos 16x}{3-4\cos 2x+\cos 4x}$$

28. Prove that the sum of the infinite series whose rth term is

$$(-1)^{r-1}\frac{r}{2r-1}\,\frac{1}{(2r-2)!}\text{ is }\frac{1}{\sqrt{2}}\sin(\tfrac{1}{4}\pi+1).$$

29. If e be very small, and $\phi=\theta-2e\sin\theta+\frac{3}{4}e^2\sin 2\theta$, shew that

$$\theta=\phi+2e\sin\phi+\tfrac{5}{4}e^2\sin 2\phi,\text{ nearly.}$$

30. If $y=z+k\sin(z+ka)$, expand z in powers of the small quantity k, as far as the term in k^4.

31. From the trigonometrical identity

$$\sin(d-b)\sin(a-c)+\sin(b-c)\sin(a-d)+\sin(c-d)\sin(a-b)=0,$$

deduce the algebraical identity

$$(d-b)(a-c)\{(d-b)^2+(a-c)^2\}+(b-c)(a-d)\{(b-c)^2+(a-d)^2\}$$
$$+(c-d)(a-b)\{(c-d)^2+(a-b)^2\}=0.$$

32. Prove that ϕ differs from $\frac{3\sin 2\phi}{2(2+\cos 2\phi)}$ by $\frac{4}{45}\phi^5$ nearly, ϕ being a small angle. (Snellius' formula.)

33. Find the circular measure, to five places of decimals, of the smallest angle which satisfies the equation $\sin(x+\frac{1}{6}\pi)=10\sin x$.

34. Solve the equation $(\sin\theta)^{a\cos\theta}=b$, approximately, where a is positive and not large, and θ is known to be nearly equal to a, which is itself not very small.

35. Shew that there is only one positive value of θ such that $\theta=2\sin\theta$, and find its value to two places of decimals by means of a table of logarithms.

36. In the relation $a\sin^{-1}x=b\sin^{-1}y$, where a and b are integers prime to each other, prove that there are $2b$ values of y for each value of x, unless a and b are both odd numbers when there are b values.

37. Assuming that if a be the acute angle whose sine is $\frac{\sqrt{3}}{4}$, $\sin 7a$ must be $\frac{\sqrt{3}}{256}$, prove that $\cos a-\cos\frac{\pi}{7}$ exceeds $\frac{3}{7.2^{10}}$ by less than ·0000005.

CHAPTER IX.

TRIGONOMETRICAL TABLES.

101. IN order that the formulae of Trigonometry may be of practical use in the solution of triangles and in other numerical calculations, it is necessary that we should possess numerical tables giving the circular functions of angles, so that from these tables we can find to a sufficient degree of accuracy the functions corresponding to a given angle, and conversely the angle corresponding to a given function. Such tables are of two kinds, (1) tables of natural[1] sines, cosines, tangents, &c., in which the numerical values of the sines, cosines, tangents, &c., of angles are given to a certain number of places of decimals, and (2) tables of logarithmic sines, cosines, tangents, &c., in which the logarithms to the base 10, of these functions, are given to a certain number of places of decimals. The latter kind of tables are those which are now used for most practical purposes; in nearly all such tables the logarithms are all increased by 10, so that the use of negative logarithms is avoided; the logarithms so increased are called tabular logarithms and written thus, $L \sin 30°$; so that $L \sin 30° = 10 + \log \sin 30°$.

Calculation of tables of natural circular functions.

102. We shall first shew how to calculate tables of the natural circular functions, which will give the values of these functions accurately to a certain specified number of places of decimals, for all angles from 0° to 90°, at certain intervals such as $1'$ or $10''$. We shall first calculate the sine and cosine of $1'$ and of $10''$.

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[1] Logarithms were formerly called "artificial" numbers, thus ordinary numbers were called "natural" numbers.

(1) *To find* sin 1′, cos 1′.

Let $\theta = \frac{\pi}{180 \times 60}$ denote the circular measure of 1′, then

$$\theta = \frac{3{\cdot}141592653589793\ldots}{10800} = {\cdot}000290888208665$$

to 15 places of decimals, hence

$$\tfrac{1}{6}\theta^3 = \tfrac{1}{6}({\cdot}0003)^3 = 000000000004$$

to 12 places of decimals.

Now from the theorem in Art. 95, sin 1′ lies between θ and $\theta - \frac{1}{6}\theta^3$, and these numbers only differ in the twelfth decimal place, therefore to eleven places of decimals

·00029088820 is the correct value of sin 1′.

We find also $1 - \frac{1}{2}\theta^2 = {\cdot}999999957692025029$

to 18 decimal places,

and $\quad \frac{1}{24}\theta^4 = \frac{1}{24}({\cdot}00029\ldots)^4 = {\cdot}00000000000000029$

to 17 decimal places.

Now cos 1′ lies between $1 - \frac{1}{2}\theta^2$ and $1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$; and since these two numbers differ only in the 16th decimal place, we have cos 1′ = ·999999957692025 correct to 15 decimal places.

(2) *To find* sin 10″, cos 10″.

If $\quad \theta = \frac{\pi}{64800}$, the circular measure of 10″,

we find $\quad \theta = {\cdot}000048481368110$, to 15 decimal places,

$\frac{1}{6}\theta^3 = {\cdot}000000000000021$, to 15 decimal places,

hence the two numbers θ and $\theta - \frac{1}{6}\theta^3$ agree to 12 decimal places, therefore sin 10″ = ·000048481368, to 12 decimal places.

Also $\frac{1}{24}\theta^4$ is zero to 17 decimal places, thus $\cos 10'' = 1 - \frac{1}{2}\theta^2$, or cos 10″ = ·9999999988248, to 13 decimal places.

103. The formulae

$$\sin nA = 2 \cos A \sin (n-1) A - \sin (n-2) A,$$
$$\cos nA = 2 \cos A \cos (n-1) A - \cos (n-2) A,$$

enable us to calculate the sines and cosines of multiples of 1′, or of 10″. Let $A = 10''$, $2 \cos 10'' = 2 - k$ where $k = {\cdot}0000000023504$, then the formulae may be written

$$\sin nA - \sin(n-1)A = \{\sin(n-1)A - \sin(n-2)A\} - k\sin(n-1)A,$$
$$\cos nA - \cos(n-1)A = \{\cos(n-1)A - \cos(n-2)A\} - k\cos(n-1)A;$$

if in these formulae we put $n=2$, we can calculate $\sin 20''$ and $\cos 20''$. We can now by letting $n=3, 4, 5, \ldots$ calculate the differences $\sin nA - \sin(n-1)A$, $\cos nA - \cos(n-1)A$, when the preceding differences $\sin(n-1)A - \sin(n-2)A$, $\cos(n-1)A - \cos(n-2)A$, and also $\sin(n-1)A$, $\cos(n-1)A$, have been found; hence these differences can be found by a continued use of the formulae; we can then find $\sin nA$, $\cos nA$, and thus we can form a table of sines and cosines of angles at intervals of $10''$. We have $k = \cdot 000000002354$, thus in calculating $k\sin(n-1)A$, $k\cos(n-1)A$ we need only use the first few figures of the value of $\sin(n-1)A$, $\cos(n-1)A$.

104. When $\sin nA$, $\cos nA$ are thus calculated by successive applications of the formulae, the errors arising from the use of approximate values of $\sin A$, $\cos A$ will accumulate during the process; it is therefore necessary to consider how many places of decimals must be used during the process, in order that with assumed values of $\sin A$, $\cos A$, correct to a certain number of places of decimals, we may obtain values of $\sin nA$, $\cos nA$ which will be correct to a prescribed number of places of decimals.

Suppose m the number of places of decimals to which $\sin A$, $\cos A$ have been calculated, and suppose that r is the number of places of decimals that is retained in the calculation of the sines and cosines of successive multiples; let u_n be the value of $\sin nA$ or $\cos nA$, obtained by this process, and $u_n + x_n$ the corresponding correct value, we have then

$$u_n + x_n = (2-k)(u_{n-1} + x_{n-1}) - (u_{n-2} + x_{n-2}),$$

also $u_n = (2-k')u_{n-1} - u_{n-2}$, where k' is the approximate value of k to r places of decimals; let $(k-k')u_{n-1} = y_n$, we have then

$$u_n = (2-k)u_{n-1} - u_{n-2} + y_n,$$

hence $$x_n = (2-k)x_{n-1} - x_{n-2} - y_n$$

or $$x_n = 2x_{n-1} - x_{n-2} - z_n, \text{ where } z_n = y_n + kx_{n-1};$$

this may be written $$(x_n - x_{n-1}) = (x_{n-1} - x_{n-2}) - z_n,$$

whence $$(x_{n-1} - x_{n-2}) = (x_{n-2} - x_{n-3}) - z_{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_2 - x_1 = x_1 - z_2;$$

therefore $$x_n - x_{n-1} = x_1 - (z_2 + z_3 + \ldots + z_n);$$

the number kx_{n-1} is very small compared with $2x_{n-1}$, hence $y_n + kx_{n-1}$ differs insensibly from y_n, hence each of the numbers $z_2, z_3 \ldots z_n$ is less than $1/10^r$, therefore their arithmetic mean θ_n is less than $1/10^r$, thus

$$x_n - x_{n-1} = x_1 - (n-1)\theta_n,$$

$$x_{n-1} - x_{n-2} = x_1 - (n-2)\theta_{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_2 - x_1 = x_1 - \theta_2,$$

or $$x_n = nx_1 - (\theta_2 + 2\theta_3 + \ldots + \overline{n-1}\,\theta_n);$$

now θ_2, θ_3 θ_n are each numerically less than $1/10^r$, hence

$$-(\theta_2+2\theta_3+\dots)$$

is less than $\frac{1}{2}n(n-1)/10^r$, or

$$x_n<\frac{n}{10^m}+\frac{n(n-1)}{2\ 10^r};$$

a fortiori
$$x_n<\frac{n}{10^m}+\frac{n^2}{2\,.\,10^r}\ \dots\dots\dots\dots\dots\dots\dots\dots\ (a).$$

If in this formula $m=12$, $n=10800$,

$$x_n<\frac{108}{10^{10}}+\frac{5832}{10^{r-4}}$$

$$<\cdot 0000000108+\cdot 00\dots\dots 5832,$$

where there are $r-8$ zeros in the last decimal, hence if $r=15$, $x_n<\cdot 00000007$, or u_n is correct to seven places of decimals; now $10800\times 10''=30^\circ$, hence the sine or cosine of 30° will be found correct to seven places of decimals if when calculating the sines or cosines of the multiples of 10″ up to 30° we retain 15 places of decimals throughout the calculation. The formula (*a*) may be applied in all such cases to determine the number r, so that x_n may be zero to a certain number of decimal places[1].

EXAMPLE.

Prove that in order to calculate the sines and cosines of multiples of 10″ up to 45°, correct to 8 places of decimals, the values of sin 10″, cos 10″ being known to 12 decimal places, it is necessary to retain 17 decimal places in the calculation.

105. When a table of sines and cosines of angles at intervals of 10″, or of 1′, is required, it is only necessary to calculate the values for angles up to 30°, we can then obtain the values of the sines and cosines of angles from 30° to 60°, by means of the formulae

$$\sin(30^\circ+A)+\sin(30^\circ-A)=\cos A,$$
$$\cos(30^\circ-A)-\cos(30^\circ+A)=\sin A,$$

by giving A all values up to 30°. When the sines and cosines of the angles up to 45° have been obtained, those of angles between 45° and 90° are obtained from the fact that the sine of an angle is equal to the cosine of its complement, so that it is unnecessary to proceed in the calculation further than 45°.

The method of calculating Tables of circular functions, which we have explained, is substantially that of Rheticus (1514—1576); his tables of sines, tangents, and secants were published in 1596, after his death The earliest table is the Table of chords in Ptolemy's *Almagest*, for angles at intervals of half a degree. Historical information on the subject of Tables will be found in Hutton's *History of Mathematical Tables*; see also De Morgan's Article on Tables in the *English Encyclopaedia*.

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[1] This article has been taken substantially from Serret's *Trigonometry*.

The verification of numerical values.

106. It is necessary to have methods of verifying the correctness of the values of the sines and cosines of angles calculated by the preceding method; this may be done by the following means:

(1) We have formed in Art. 66 a table of the surd values of the sines and cosines of the angles 3°, 6°, 9° ... differing by 3°; we can therefore calculate the sines and cosines of these angles to any required number of places of decimals, then the values of the functions obtained by the method of calculation above explained may be compared with the values thus obtained. If necessary, the values of the sines and cosines of angles differing by $1^\circ\,30'$ may be obtained by means of the dimidiary formulae, and we have thus a still closer check upon the calculations.

(2) There are certain well-known formulae called formulae of verification, these are

$$\cos(36^\circ+A)+\cos(36^\circ-A)=\cos A+\sin(18^\circ+A)+\sin(18^\circ-A),$$

$$\sin A=\sin(36^\circ+A)-\sin(36^\circ-A)+\sin(72^\circ-A)-\sin(72^\circ+A)$$

(Euler's formulae),

$$\cos A=\sin(54^\circ+A)+\sin(54^\circ-A)-\sin(18^\circ+A)-\sin(18^\circ-A)$$

(Legendre's formula).

The verification consists in the substitution of the values obtained of the functions in these identities.

Tables of tangents and secants.

107. To form a table of tangents, we find the tangents of angles up to 45° from the tables of sines and cosines by means of the formula $\tan A=\sin A/\cos A$; the tangents of angles from 45° to 90° may then be obtained by means of Cagnoli's formula

$$\tan(45^\circ+A)=2\tan 2A+\tan(45^\circ-A).$$

A table of cosecants can be formed by means of the formula $\operatorname{cosec} A=\tan\frac{1}{2}A+\cot A$, and a table of secants by means of the formula $\sec A=\tan A+\tan(45^\circ-\frac{1}{2}A)$.

Calculation by series.

108. A more modern method of calculating the sines and cosines of angles is to use the series in Art. 99; if we put $x=\frac{m}{n}\cdot\frac{\pi}{2}$ we have

$$\sin\left(\frac{m}{n}\,90^\circ\right)=\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)-\frac{1}{3\,!}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^3+\frac{1}{5\,!}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^5-\ldots,$$

$$\cos\left(\frac{m}{n}\,90^\circ\right)=1-\frac{1}{2\,!}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^2+\frac{1}{4\,!}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)^4-\ldots.$$

We thus obtain the formulae

$\sin(m/n\,90^\circ)=$	1·57079	63267	94896	61923	13	m/n
	−0·64596	40975	06246	25365	58	m^3/n^3
	+0·07969	26262	46167	04512	05	m^5/n^5
	−0·00468	17541	35318	68810	07	m^7/n^7
	+0·00016	04411	84787	35982	19	m^9/n^9
	−0·00000	35988	43235	21208	53	m^{11}/n^{11}
	+0·00000	00569	21729	21967	93	m^{13}/n^{13}
	−0·00000	00006	68803	51098	11	m^{15}/n^{15}
	+0·00000	00000	06066	93573	11	m^{17}/n^{17}
	−0·00000	00000	00043	77065	47	m^{19}/n^{19}
	+0·00000	00000	00000	25714	23	m^{21}/n^{21}
	−0·00000	00000	00000	00125	39	m^{23}/n^{23}
	+0·00000	00000	00000	00000	52	m^{25}/n^{25}
$\cos(m/n\,90^\circ)=$	1·00000	00000	00000	00000	00	
	−1·23370	05501	36169	82735	43	m^2/n^2
	+0·25366	95079	01048	01363	66	m^4/n^4
	−0·02086	34807	63352	96087	31	m^6/n^6
	+0·00091	92602	74839	42658	02	m^8/n^8
	−0·00002	52020	42373	06060	55.	m^{10}/n^{10}
	+0·00000	04710	87477	88181	72	m^{12}/n^{12}
	−0·00000	00063	86603	08379	19	m^{14}/n^{14}
	+0·00000	00000	65659	63114	98	m^{16}/n^{16}
	−0·00000	00000	00529	44002	01	m^{18}/n^{18}
	+0·00000	00000	00003	43773	92	m^{20}/n^{20}
	−0·00000	00000	00000	01835	99	m^{22}/n^{22}
	+0·00000	00000	00000	00008	21	m^{24}/n^{24}
	−0·00000	00000	00000	00000	03	m^{26}/n^{26}

Since we need only calculate the sines and cosines of angles up to 45°, the fraction m/n is always taken less than $\frac{1}{2}$, so that very few terms of the series suffice for the calculation to a small number of decimal places. These series are taken from Euler's *Analysis of the Infinite*, where they are given to six more decimal places.

Logarithmic tables.

109. When tables of natural sines and cosines have been constructed, tables of logarithmic sines and cosines may be made by means of tables of ordinary logarithms which will give the logarithm of the calculated numerical value of the sine or cosine of any angle; adding 10 to the logarithm so found, we have the corresponding tabular logarithm. The logarithmic tangents may be found by means of the relation $L \tan A = 10 + L \sin A - L \cos A$, and thus a table of logarithmic tangents may be constructed. We shall in a later Chapter give a direct method by which tables of logarithmic sines, cosines, and tangents may be constructed.

Description and use of trigonometrical tables.

110. Trigonometrical tables, either natural or logarithmic, are constructed as follows:—

(1) They give directly the functions for angles between 0° and 90° only; the values of the functions for angles of magnitudes beyond these limits may be at once deduced.

(2) The tables give the values of the functions of angles from 0° to 45°, and from 45° to 90°, by means of a double reading of the same figures; the names of the functions, sine, cosine, tangent, and also the degrees ($< 45°$), are printed at the top of the page, and the corresponding minutes and seconds are printed in the left-hand column, the angles increasing as we go down the page; again the names cosine, sine, cotangent, &c., and the degrees ($> 45°$), are printed at the bottom of the page, in the same columns in which sine, cosine, tangent, respectively are printed at the top; in the right-hand column are printed the minutes and seconds for the angles which are complementary to the former ones, these latter angles of course increasing as we go

up the page. We give as a specimen a portion of a page of Callet's seven-figure logarithmic tables for angles at intervals of 10″.

17 deg.

′	″	sine	dif.	cosine	dif.	tangent	dif.	cotangent	″	′
50	0	9·4860749		9 9786148		9·5074602		0·4925398	0	10
			655		68		722			
	10	9·4861404		9·9786080		9·5075324		0·4924676	50	
			654		68		722			
	20	9·4862058		9·9786012		9·5076046		0·4923954	40	
			654		68		722			
	30	9·4862712		9·9785944		9 5076768		0·4923232	30	
			654		67		722			
	40	9·4863366		9·9785877		9·5077490		0·4922510	20	
			654		68		722			
	50	9·4864020		9·9785809		9·5078212		0·4921788	10	
			654		68		721			
51	0	9·4864674		9·9785741		9·5078933		0·4921067	0	9
			654		68		722			
	10	9·4865328		9·9785673		9·5079655		0·4920345	50	
			654		68		721			
	20	9·4865982		9·9785605		9·5080376		0·4919624	40	
			653		67		722			
	30	9·4866635		9·9785538		9·5081098		0·4918902	30	
			654		68		721			
	40	9·4867289		9·9785470		9 5081819		0·4918181	20	
			653		68		721			
	50	9·4867942		9·9785402		9·5082540		0·4917460	10	
			653		68		721			
52	0	9·4868595		9·9785334		9·5083261		0·4916739	0	
′	″	cosine جیب متمم	dif. فرق	sine جیب	dif.	cotangent ظل متمم	dif. فرق	tangent ظل	″	′

72 deg.

For example, in the third line of the column headed cosine, we find that 9·9786012 is the tabular logarithmic cosine of the angle 17° 50′ 20″, and reading the minutes and seconds in the right-hand column we see that the same number is the logarithmic sine of the complementary angle 72° 9′ 40″. It should be observed that the logarithmic sines and tangents increase with the angle, whereas the logarithmic cosines and cotangents diminish with the angle.

111. In order to find the functions corresponding to an angle whose magnitude lies between two of the angles for which the functions are tabulated, we use the principle which we shall presently investigate that, *except for angles which are either very small or very nearly equal to a right angle, small changes in the natural or in the logarithmic function of an angle are proportional to the change in the angle itself.*

For example, if the difference between two consecutive tabulated values corresponding to a difference of 10″ in the angle is a, the difference between the values of the function for the smaller tabular angle and an angle greater than this angle by y'' is $\frac{y}{10}a$; the increase of the function for an increase $10''$ of the angle is a, and that for an increase y'' ($< 10''$) is that fraction of a which y'' is of $10''$. In the specimen of Callet's tables which we have given, the differences between consecutive logarithms are given without the decimal points in the columns headed *dif.*

For example, suppose we wish to find $L\sin 17^\circ\ 51'\ 13''$, we find from the table

$$L\sin 17^\circ\ 51'\ 10'' = 9{\cdot}4865328,$$
$$L\sin 17^\circ\ 51'\ 20'' = 9{\cdot}4865982,$$
$$\textit{dif.} = 654\,;$$

we have $\frac{3}{10} \times 654 = 196{\cdot}2$, hence we must add ·0000196 to the first logarithm and we obtain $L\sin 17^\circ\ 51'\ 13'' = 9{\cdot}4865522$

Again suppose we require the angle whose tabular logarithmic tangent is 9·5082032. We find from the table that the given logarithm lies between the two

$$L\tan 17^\circ\ 51'\ 40'' = 9{\cdot}5081819,$$
$$L\tan 17^\circ\ 51'\ 50'' = 9{\cdot}5082540,$$
$$\textit{dif.} = 721\,;$$

the difference between the given logarithmic tangent and the first obtained from the table is 213, hence the angle to be added to $17^\circ\ 51'\ 40''$ is $\frac{213}{721} \times 10'' = 2''{\cdot}9$ approximately, hence the required angle is $17^\circ\ 51'\ 43''$ approximately.

The principle of proportional parts.

112. We shall now investigate how far, and with what exceptions, the principle of proportional increase, which we have assumed in the last Article, is true.

Suppose x to denote any angle, and $f(x)$ to denote a natural or logarithmic function of x, we shall shew in the various cases that if h be any small angle measured in circular measure, added to x,

$$f(x+h) - f(x) = hf'(x) + h^2R,$$

where $f'(x)$ is another function of x, and R is a function which remains finite when $h = 0$. From this we see that, provided h be sufficiently small, $f(x+h) - f(x)$ is for a given value of x proportional to h, and it will appear that in general h^2R will be so small that it will not affect the values of the functions to the number of decimal places to which they are tabulated; hence $\dfrac{f(x+h)-f(x)}{h}$ is constant to the requisite number of decimal places for a given value of x. However, two exceptional cases will arise.

(1) If x be such that $f'(x)$ is very small, then the difference $f(x+h)-f(x)$ may vanish, to the order in the tables; the difference $f(x+h)-f(x)$ is then said to be *insensible*, and in that case two or more consecutive tabulated values of $f(x)$ may be the same.

(2) If x is such that R is large compared with $f'(x)$, the term h^2R may not be small compared with $hf'(x)$; in this case the difference $f(x+h)-f(x)$ is not proportional to h, and is said to be *irregular*.

In either of these cases (1) and (2) the method of proportions fails, but we shall shew how by special devices the difficulties are obviated.

The student who is acquainted with Taylor's theorem will see that the formula given above is really the special case of Taylor's theorem

$$f(x+h)=f(x)+hf'(x)+\tfrac{1}{2}h^2f''(x+\theta h),$$

where θ is between 0 and 1, thus $R=\frac{1}{2}f''(x+\theta h)$, and the error made in assuming $f(x+h)-f(x)=hf'(x)$ lies between the greatest and least values which $\frac{1}{2}h^2f''(z)$ assumes between the limits $z=x$ and $z=x+h$.

113. First let $f(x)=\sin x$,

then $$\sin(x+h)=\sin x\cos h+\cos x\sin h,$$

or $$\sin(x+h)-\sin x=\cos x\left(h-\tfrac{1}{6}h^3+\ldots\right)-\sin x\left(\tfrac{1}{2}h^2-\tfrac{1}{24}h^4+\ldots\right)$$
$$=h\cos x-\tfrac{1}{2}h^2\sin x+\text{higher powers of } h;$$

in this case $f'(x)=\cos x$, and the approximate value of R is $-\frac{1}{2}\sin x$;

thus $$\sin(x+h)-\sin x=h\cos x-\tfrac{1}{2}h^2\sin x\ldots\ldots\ldots\ldots(1)$$

is the approximate difference equation.

Similarly it may be shewn that, approximately,

$$\cos(x+h)-\cos x=-h\sin x-\tfrac{1}{2}h^2\cos x\ldots\ldots\ldots(2).$$

Again $$\tan(x+h)-\tan x=\frac{\sin h}{\cos x\cos(x+h)}$$
$$=\frac{h}{\cos^2 x-h\sin x\cos x},$$

or, approximately,

$$\tan(x+h)-\tan x=h\sec^2 x+h^2\sec^2 x\tan x\ldots\ldots(3).$$

Also $L\sin(x+h) - L\sin x = \log\dfrac{\sin(x+h)}{\sin x}$

$= \log(1 - \tfrac{1}{2}h^2 + h\cot x)$,

or $L\sin(x+h) - L\sin x = h\cot x - \tfrac{1}{2}h^2\operatorname{cosec}^2 x$ ......(4)

Similarly $L\cos(x+h) - L\cos x = -h\tan x - \tfrac{1}{2}h^2\sec^2 x$......(5),

$L\tan(x+h) - L\tan x = \dfrac{h}{\sin x\cos x} - 2h^2\dfrac{\cos 2x}{\sin^2 2x}$ ...(6).

In each case we have found only the approximate value of R, that is to say, we have left out the terms involving cubes and higher powers of h. It appears from these six equations that if h is small enough, the differences are, for values of x which are neither small nor nearly equal to a right angle, proportional to h. The following exceptional cases arise.

(1) The difference $\sin(x+h) - \sin x$ is insensible when x is nearly a right angle, for in that case $h\cos x$ is very small; it is then also irregular, for $\tfrac{1}{2}h^2\sin x$ may become comparable with $h\cos x$.

(2) The difference $\cos(x+h) - \cos x$ is insensible when x is small; it is then also irregular.

(3) The difference $\tan(x+h) - \tan x$ is irregular when x is nearly a right angle, for $h^2\sec^2 x\tan x$ may then become comparable with $h\sec^2 x$.

(4) The difference $L\sin(x+h) - L\sin x$ is irregular when x is small, and both insensible and irregular when x is nearly a right angle.

(5) The difference $L\cos(x+h) - L\cos x$ is insensible and irregular when x is small, and irregular when x is nearly a right angle.

(6) The difference $L\tan(x+h) - L\tan x$ is irregular when x is either small or nearly a right angle.

It should be noticed that a difference which is insensible is also irregular, but that the converse does not hold.

In order to investigate the degree of approximation to which the principle of proportional parts is in any case true, it is the simplest way to consider the true value of R; in the case of $\sin(x+h) - \sin x$ the true value of the second term is $-\tfrac{1}{2}h^2\sin(x+\theta h)$, where θ is between 0 and 1; if the table is for intervals of 10″, the greatest value of $\tfrac{1}{2}h^2$ is $\tfrac{1}{2}\left(\dfrac{10\pi}{60\times 60\times 180}\right)^2$ or $\tfrac{1}{2}(\cdot 00005)^2$;

this gives no error in the first eight places of decimals; in the case of $\tan(x+h)-\tan x$ the error is $(\cdot 00005)^2 \sec^2(x+\theta h)\tan(x+\theta h)$; hence when $\tan x+\tan^3 x=40$, the error will begin to appear in the seventh place of decimals. In the case of $L\sin x$ there is no error in the seventh place of decimals if $x>5°$

114. When the differences for a function are insensible to the number of decimal places of the tables, the tables will give the function when the angle is known, but we cannot employ the tables to find any intermediate angle by means of this function; thus we cannot determine x from the value of $L\cos x$, for small angles, or from the value of $L\sin x$, for angles nearly equal to a right angle. When the differences for a function are irregular without being insensible, the approximate method of proportional parts is not sufficient for the determination of the angle by means of the function, nor the function by means of the angle, thus the approximation is inadmissible for $L\sin x$, when x is small, for $L\cos x$ when x is nearly a right angle, and for $L\tan x$ in either case.

In these cases of irregularity without insensibility, the following means may be used to effect the purpose of finding the angle corresponding to a given value of the function, or of the function corresponding to a given angle

(1) We may use tables of $L\sin x$, $L\tan x$ for the first few degrees calculated for angles at intervals of one second, and for $L\cos x$, $L\tan x$ for the few degrees near 90°, calculated for each second; Callet gives such a table in his trigonometrical tables; we can then use the principle of proportional parts for all angles which are not extremely near zero or a right angle.

(2) *Delambre's method.*

This method consists of splitting $L\sin x$ or $L\tan x$ into the sum of two terms, the differences for one of which are insensible for values of x near those at which the irregularity takes place, and the differences for the other one are regular; the difference for the first of these terms is irregular, but this is of no consequence, owing to its being also insensible. Thus if x be the circular measure of n'' a small angle,

$$L\sin n''=\left(\log\frac{\sin x}{x}+La\right)+\log n,$$

$$L \tan n'' = \left(\log \frac{\tan x}{x} + La\right) + \log n,$$

where a is the circular measure of 1″.

Now $$\log (n+h) - \log n = \log \left(1 + \frac{h}{n}\right)$$
$$= \frac{h}{n} - \frac{h^2}{2n^2} + \dots;$$

hence the differences for $\log n$ are regular, if h be small compared with n. Also the differences for $\log \frac{\sin x}{x}$, $\log \frac{\tan x}{x}$ are insensible, for

$$\log \frac{\sin (x+h)}{x+h} - \log \frac{\sin x}{x} = \log \frac{\sin (x+h)}{\sin x} - \log \frac{x+h}{x}$$
$$= h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x - \frac{h}{x} + \frac{h^2}{2x^2}$$
$$= h\left(\cot x - \frac{1}{x}\right) + \frac{h^2}{2}\left(\frac{1}{x^2} - \operatorname{cosec}^2 x\right)$$

and $$\log \frac{\tan (x+h)}{x+h} - \log \frac{\tan x}{x}$$
$$= h\left(\frac{1}{\sin x \cos x} - \frac{1}{x}\right) + \frac{h^2}{2}\left(-\frac{4 \cos 2x}{\sin^2 2x} + \frac{1}{x^2}\right);$$

each of these differences is insensible since the coefficient of h is small when x is small.

If tables of the values of $\log \frac{\sin x}{x} + La$, $\log \frac{\tan x}{x} + La$ are constructed for the first few degrees of the quadrant, we may use these tables in conjunction with the tables of natural logarithms of numbers to find n accurately when $L \sin n''$ or $L \tan n''$ is given, and conversely.

If $L \sin n''$ or $L \tan n''$ is given, find the *approximate* value of n; then from the table we get the value of $\log \frac{\sin x}{x} + La$ or $\log \frac{\tan x}{x} + La$, either of which changes very slowly; then $\log n$ is given by the value

$$L \sin n'' - \left(\log \frac{\sin x}{x} + La\right),$$

or $$L \tan n'' - \left(\log \frac{\tan x}{x} + La\right),$$

and we find n accurately from the table of natural logarithms. If n is given, the table gives the value of $\log\frac{\sin x}{x} + La$, and $\sin n''$ is then found by the formula.

(3) *Maskelyne's method.*

The principle of this method is the same as that of Delambre's. If x is a small angle, we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} = \left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}} = \cos^{\frac{1}{3}} x, \text{ approximately};$$

hence
$$\log \sin x = \log x + \tfrac{1}{3} \log \cos x;$$

when x is a small angle the differences of $\log \cos x$ are insensible, hence it is sufficient to use an approximate value of $\cos x$. If $\log \sin x$ is given we find an approximate value of x, and use that for finding $\log \cos x$; x is then obtained from the above equation. If x is given we can find $\log x$ accurately from the table of natural logarithms, and also an approximate value of $\log \cos x$; the formula then gives $\log \sin x$. We can shew, in a similar manner, that $\log \tan x$ is given by the formula $\log \tan x = \log x - \frac{2}{3} \log \cos x$.

EXAMPLE.

Shew that the following formula is more nearly true than Maskelyne's :—

$$\log \sin \theta = \log \theta - \tfrac{1}{45} \log \cos \theta + \tfrac{64}{45} \log \cos \tfrac{1}{2}\theta.$$

Adaptation of formulae to logarithmic calculation.

115. In order to reduce an expression to a form in which the numerical values can be calculated from tables of logarithms, we must make such substitutions as will reduce the given expression to the product of simple expressions; this may be frequently done by means of one or more subsidiary angles, as the following examples will shew.

(1) $\sqrt[3]{a^6 + b^6} = a^2 \sec^{\frac{2}{3}} \phi$, where $\tan \phi = b^3/a^3$; hence

$$\log \sqrt[3]{a^6 + b^6} = 2 \log a + \tfrac{2}{3} (L \sec \phi - 10),$$

where
$$L \tan \phi = 10 + 3 (\log b - \log a);$$

thus $\sqrt[3]{a^6 + b^6}$ can be calculated by means of logarithmic tables, ϕ having first been found from the tables.

(2) $a\cos\alpha + b\sin\alpha = a\cos(\alpha - \phi)\sec\phi$, where $\tan\phi = b/a$; hence

$$\log(a\cos\alpha + b\sin\alpha) = \log a + L\cos(\alpha - \phi) - L\cos\phi,$$

where ϕ is found from

$$L\tan\phi = 10 + \log b - \log a.$$

116. To calculate numerically the roots of a quadratic equation supposing the roots to be real.

Let $ax^2 + bx + c = 0$ be the equation, and first suppose a and c to be both positive. We have $\tan^2\theta - 2\operatorname{cosec} 2\theta \tan\theta + 1 = 0$; now let $x = y\sqrt{c/a}$, the equation becomes $y^2 + by/\sqrt{ac} + 1 = 0$; hence if $\sin 2\theta = 2\sqrt{ac}/b$, the quadratic in y will be the same as that in $-\tan\theta$, the roots of which are $-\tan\theta$, $-\cot\theta$; thus the roots of the given quadratic are $-\sqrt{c/a}\tan\theta$, $-\sqrt{c/a}\cot\theta$, where $\sin 2\theta = 2\sqrt{ac}/b$, and hence the roots may be calculated by means of logarithmic tables.

If a and c are of opposite signs, we may take the quadratic to be $ax^2 + bx - c = 0$; in this case put $x = y\sqrt{c/a}$ and it reduces to $y^2 + by/\sqrt{ac} - 1 = 0$; comparing this with the equation

$$\tan^2\theta + 2\cot 2\theta \tan\theta - 1 = 0$$

we see that if $\tan 2\theta = 2\sqrt{ac}/b$, the roots of the quadratic in x are $\sqrt{c/a}\tan\theta$ and $-\sqrt{c/a}\cot\theta$.

117. To calculate the roots of the cubic $x^3 + qx + r = 0$ supposing them all to be real. We shall suppose q to be negative.

Consider the equation

$$\sin^3\theta - \tfrac{3}{4}\sin\theta + \tfrac{1}{4}\sin 3\theta = 0;$$

let $x = y\sqrt{-4q/3}$, then the equation in x becomes

$$y^3 - \tfrac{3}{4}y + r(-3/4q)^{\frac{3}{2}} = 0;$$

this will be the same as the cubic in $\sin\theta$, if

$$\sin 3\theta = 4r(-3/4q)^{\frac{3}{2}} = (-27r^2/4q^3)^{\frac{1}{2}};$$

hence the values of x are

$$\sqrt{-4q/3}\sin\theta,\quad \sqrt{-4q/3}\sin(\theta + \tfrac{2}{3}\pi),\quad \sqrt{-4q/3}\sin(\theta + \tfrac{4}{3}\pi),$$

the condition that $\sin 3\theta \ngtr 1$ is the condition that the roots of the cubic are all real.

We shall shew in a later Chapter how to calculate the roots of a cubic when two of them are imaginary.

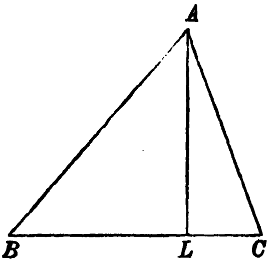
The processes by which we have solved the quadratic and cubic equations shew that the two algebraical problems are really equivalent to the geometrical problems of bisecting and trisecting an angle respectively. It follows that a quadratic equation can be solved graphically by means of the ruler and compasses only, whereas the cubic can not in general be solved graphically by these means, since they are inadequate for solving generally the geometrical problem of trisecting an angle.

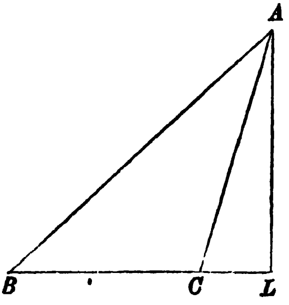
CHAPTER X

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

118. IF ABC be any triangle, we shall denote the magnitudes of the angles BAC, ABC, ACB by A, B, C respectively, and the lengths of the sides BC, CA, AB by a, b, c respectively. We shall, in this Chapter, investigate various important formulae connecting the sides a, b, c of a triangle with the circular functions of the angles. These formulae will afford the basis of the methods by which we shall solve a triangle in the various cases in which three parts of the triangle are given.

119. From the fundamental theorem in projections we see that the sum of the projections of BA, AC, on BC, is equal to BC, and that the sum of their projections on a perpendicular to BC is zero. Expressing these facts we have, since the positive direction of AC makes an angle $-C$ with the positive direction of BC,





$$BA\cos B + AC\cos C = a,$$

or
$$c\cos B + b\cos C = a,$$

and
$$BA\sin B - AC\sin C = 0, \quad \text{or} \quad c\sin B - b\sin C = 0,$$

which may be written $b/\sin B = c/\sin C$. These relations and the corresponding ones obtained by projecting on and perpendicular to each of the other sides, in turn, may be written

$$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$a/\sin A = b/\sin B = c/\sin C \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equations (2) express the fact that, *in any triangle, the sides are proportional to the sines of the opposite angles.*

120. The relations (2) may also be proved thus:—Draw the circle circumscribing the triangle ABC, and let R be the length of its radius, then the side BC is equal to twice the radius multiplied by the sine of half the angle BC subtends at the centre of the circle, that is

$$BC = 2R\sin A \quad \text{or} \quad 2R\sin(180° - A),$$

hence $a = 2R\sin A$; similarly

$$b = 2R\sin B, \text{ and } c = 2R\sin C,$$

hence $$a/\sin A = b/\sin B = c/\sin C = 2R.$$

These relations (2) may also be deduced from (1); writing the first two equations (1) in the form

$$a - b\cos C - c\cos B = 0,$$

$$-a\cos C + b - c\cos A = 0,$$

we can determine the ratios of a, b, c; we obtain

$$\frac{a}{\cos C\cos A + \cos B} = \frac{b}{\cos B\cos C + \cos A} = \frac{c}{1 - \cos^2 C};$$

hence $$\frac{a}{\sin A\sin C} = \frac{b}{\sin B\sin C} = \frac{c}{\sin^2 C}, \text{ or } a/\sin A = b/\sin B = c/\sin C.$$

To deduce (1) from (2) we have

$$a = \frac{a}{\sin A}\sin(B + C) = \frac{a}{\sin A}(\sin B\cos C + \cos B\sin C);$$

hence $$a = \frac{b}{\sin B}\sin B\cos C + \frac{c}{\sin C}\cos B\sin C = b\cos C + c\cos B,$$

which is the first of the relations (1).

If we eliminate a, b, c from the three equations in (1), we obtain the relation $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$, which holds between the cosines of the angles of a triangle.

121. If we multiply the equations in (1) by $-a$, b, c respectively, and then add, we have

$$b^2+c^2-a^2=2bc\cos A,$$

which gives an expression for the cosine of an angle, in terms of the sides; we may write this relation and the two similar ones for $\cos B$, $\cos C$ thus

$$\left.\begin{aligned} a^2&=b^2+c^2-2bc\cos A\\ b^2&=c^2+a^2-2ca\cos B\\ c^2&=a^2+b^2-2ab\cos C\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

122. We may obtain these relations (3) directly by means of Euclid, Bk. II. Props. 12 and 13. If AL be perpendicular to BC, we have, when C is an acute angle,

$$AB^2=AC^2+BC^2-2BC\,.\,CL,$$

and when C is obtuse

$$AB^2=AC^2+BC^2+2BC\,.\,CL;$$

in the first case $CL=AC\cos C$, and in the second case

$$CL=AC\cos(180^\circ-C)=-AC\cos C;$$

therefore in either case

$$c^2=a^2+b^2-2ab\cos C.$$

To deduce the relations (2) from (3) we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc};$$

therefore

$$\sin^2 A=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2c^2}=\frac{(2bc+b^2+c^2-a^2)(2bc+a^2-b^2-c^2)}{4b^2c^2}$$

or

$$\sin^2 A=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}.$$

thus $\dfrac{\sin^2 A}{a^2}$ is equal to the symmetrical quantity

$$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2b^2c^2};$$

hence

$$\frac{\sin^2 A}{a^2}=\frac{\sin^2 B}{b^2}=\frac{\sin^2 C}{c^2},$$

from which (2) follows.

To deduce (1) from (3), divide the first two equations of (3) by c, and then add them; we get

$$\frac{a^2+b^2}{c}=2c+\frac{a^2+b^2}{c}-2(b\cos A+a\cos B),\quad\text{or}\quad c=b\cos A+a\cos B$$

123. We have

$$\sin^2\tfrac{1}{2}A=\tfrac{1}{2}(1-\cos A),\quad \cos^2\tfrac{1}{2}A=\tfrac{1}{2}(1+\cos A);$$

hence

$$\sin^2\tfrac{1}{2}A=\tfrac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right),\quad \cos^2\tfrac{1}{2}A=\tfrac{1}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right),$$

or

$$\sin^2\tfrac{1}{2}A=\frac{(a+b-c)(a-b+c)}{4bc},\quad \cos^2\tfrac{1}{2}A=\frac{(a+b+c)(b+c-a)}{4bc}.$$

Now let $2s=a+b+c$, then $2(s-a)=b+c-a$, and we have

$$\sin^2\tfrac{1}{2}A=\frac{(s-b)(s-c)}{bc},\quad \cos^2\tfrac{1}{2}A=\frac{s(s-a)}{bc};$$

therefore

$$\sin\tfrac{1}{2}A=\left\{\frac{(s-b)(s-c)}{bc}\right\}^{\frac{1}{2}},\quad \cos\tfrac{1}{2}A=\left\{\frac{s(s-a)}{bc}\right\}^{\frac{1}{2}}.$$

$$\tan\tfrac{1}{2}A=\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}^{\frac{1}{2}}\quad\dots\dots\dots\dots\dots\dots(4);$$

these formulae are more convenient than (3) as a means of determining functions of the angles when the sides are given, because they are more easily capable of being adapted to logarithmic calculation.

124. Since $\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$, we have

$$\frac{\sin B\pm\sin C}{\sin A}=\frac{b\pm c}{a},\quad\text{or}\quad \frac{2\sin\frac{1}{2}(B\pm C)\cos\frac{1}{2}(B\mp C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)}=\frac{b\pm c}{a},$$

hence
$$\frac{b+c}{a}=\frac{\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)},\quad\text{and}\quad\frac{b-c}{a}=\frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)},$$

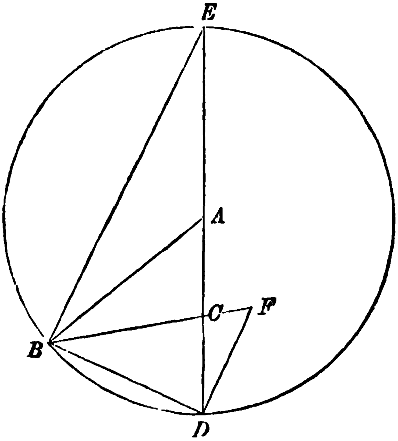
or
$$a=\frac{(b+c)\sin\frac{1}{2}A}{\cos\frac{1}{2}(B-C)},\quad a=\frac{(b-c)\cos\frac{1}{2}A}{\sin\frac{1}{2}(B-C)}\quad\dots\dots\dots(5),$$

we obtain by division the formula

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A\quad\dots\dots\dots\dots\dots(5').$$

To prove these formulae geometrically, with centre A and radius AB describe a circle cutting AC in D and E; draw DF parallel to BE, then $CE=b+c$, $DC=c-b$, $DEB=\frac{1}{2}A$, $DBF=C+\frac{1}{2}A-90^\circ=\frac{1}{2}C-\frac{1}{2}B$. We have

$$\frac{CD}{CB}=\frac{\sin DBF}{\sin CDB},\quad\text{or}\quad\frac{b-c}{a}=\frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A},$$



also $$\frac{b+c}{c-b}=\frac{CE}{CD}=\frac{EB}{DF}=\frac{BD\cot\frac{1}{2}A}{BD\tan\frac{1}{2}(C-B)}=\frac{\cot\frac{1}{2}A}{\tan\frac{1}{2}(C-B)};$$

hence $$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A.$$

The area of a triangle.

125. The area of a triangle is half that of a parallelogram on the same base and with the same altitude; if the side a is the base, the altitude is $b\sin C$ or $c\sin B$, we have thus the expressions

$$\tfrac{1}{2}ab\sin C \text{ and } \tfrac{1}{2}ac\sin B$$

for the area of the triangle; the area of a triangle is therefore *half the product of any two sides multiplied by the sine of the included angle.*

Using the expression for $\sin A$, found in Art. 122,

$$\frac{1}{2bc}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)},$$

we have for the area of a triangle the expression

$$\tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)},$$

or $$\sqrt{s(s-a)(s-b)(s-c)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

this formula was obtained by Hero of Alexandria[1] (about 125 B.C.)

The formula (6) may also be written

$$\tfrac{1}{4}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}.$$

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[1] See Ball's *History of Mathematics*, p. 82, where the original geometrical proof of the formula is given.

Variations in the sides and angles of a triangle.

126. We shall now investigate the relations which hold between small positive or negative increments in the values of the sides and angles of a triangle. Suppose three of the parts of a triangle to have been measured, of which one at least is a side, the other three parts will be determined by means of the formulae of this Chapter; the relations between the increments of the parts will enable us to find the effect in producing errors in the values of the latter three parts of small inaccuracies in the measurement of the former parts. We shall suppose that the increments are so small that their squares and products may be neglected.

Suppose A, B, C, a, b, c to be the values of the angles and sides of a triangle, as ascertained by the measurement of one side and two angles, two sides and one angle, or the three sides, the other three values being connected with the three measured ones by means of the formulae given above. If the three parts have been measured inaccurately, there will be consequent inaccuracies in the values of the other three parts as found by the formulae; let $A+\delta A$, $B+\delta B$, $C+\delta C$, $a+\delta a$, $b+\delta b$, $c+\delta c$ be the accurate values of the angles and sides; we shall obtain relations between the six errors δA, δB, δC, δa, δb, δc. It will be convenient to suppose the increments of the angles to be measured in circular measure; they can however of course be at once reduced to seconds.

We have
$$c\sin B - b\sin C = 0,$$
$$(c+\delta c)\sin(B+\delta B) - (b+\delta b)\sin(C+\delta C) = 0;$$
since, when the squares of δB, δC are neglected,
$$\sin(B+\delta B) = \sin B + \delta B\cos B, \quad \sin(C+\delta C) = \sin C + \delta C\cos C,$$
we have, $(c+\delta c)(\sin B+\delta B\cos B) - (b+\delta b)(\sin C+\delta C\cos C) = 0$

hence if we neglect the products δc, δB, δb, δC, we have
$$c\cos B\,.\,\delta B + \sin B\,.\,\delta c - b\cos C\,.\,\delta C - \sin C\,.\,\delta b = 0.$$
This, with the two corresponding equations, may be written
$$\left.\begin{aligned}\sin C\,.\,\delta b - \sin B\,.\,\delta c &= c\cos B\,.\,\delta B - b\cos C\,.\,\delta C\\ \sin A\,.\,\delta c - \sin C\,.\,\delta a &= a\cos C\,.\,\delta C - c\cos A\,.\,\delta A\\ \sin B\,.\,\delta a - \sin A\,.\,\delta b &= b\cos A\,.\,\delta A - a\cos B\,.\,\delta B\end{aligned}\right\}\ldots\ldots(7).$$

Also $$\delta A + \delta B + \delta C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$
in virtue of the relations

$$A + B + C = \pi, \quad A + \delta A + B + \delta B + C + \delta C = \pi.$$

The equations (7) are not independent, as may be seen by writing them in the form

$$\frac{\delta b}{b} - \frac{\delta c}{c} = \cot B \,.\, \delta B - \cot C \,.\, \delta C,$$

$$\frac{\delta c}{c} - \frac{\delta a}{a} = \cot C \,.\, \delta C - \cot A \,.\, \delta A,$$

$$\frac{\delta a}{a} - \frac{\delta b}{b} = \cot A \,.\, \delta A - \cot B \,.\, \delta B,$$

which shews that any one of the equations may be deduced from the other two.

The system consisting of two of the equations (7) and the equation (8) is sufficient to determine any three of the six errors when the other three are given, except that one at least of the three given errors must belong to a side.

By eliminating δB, δC, between (7) and (8), we obtain an equation giving δa in terms of δb, δc, and δA; this may however be found directly from the formula $a^2 = b^2 + c^2 - 2bc \cos A$; we obtain

$$a\delta a = (b - c\cos A)\,\delta b + (c - b\cos A)\,\delta c + bc \sin A\,\delta A,$$

which, with the two corresponding formulae, becomes, in virtue of (1),

$$\left.\begin{aligned} a\delta a &= a\cos C \,.\, \delta b + a\cos B \,.\, \delta c + bc\sin A \,.\, \delta A \\ b\delta b &= b\cos A \,.\, \delta c + b\cos C \,.\, \delta a + ca\sin B \,.\, \delta B \\ c\delta c &= c\cos B \,.\, \delta a + c\cos A \,.\, \delta b + ab\sin C \,.\, \delta C \end{aligned}\right\} \ldots\ldots (9).$$

Relations between the sides and angles of polygons.

127. Let $a_1, a_2, a_3 \ldots a_n$ denote the lengths of the sides, taken in order, of any plane closed polygon, and let $\alpha_1, \alpha_2 \ldots \alpha_n$ denote the angles, measured positively all in the same direction, which these sides make with any fixed straight line in the plane of the polygon; then from the fundamental theorem in projections in Art. 17, we have, projecting on the fixed straight line and perpendicular to it, the two relations

$$a_1 \cos\alpha_1 + a_2 \cos\alpha_2 + \ldots\ldots + a_n \cos\alpha_n = 0,$$

$$a_1 \sin\alpha_1 + a_2 \sin\alpha_2 + \ldots\ldots + a_n \sin\alpha_n = 0$$

Now let the line on which the projection is made be the side a_n; if we denote by β_1 the external angle between a_n and a_1, by β_2 the external angle between a_1 and a_2, &c., then

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_1 + \beta_2,\ \alpha_3 = \beta_1 + \beta_2 + \beta_3,\ \text{\&c.},\ \alpha_n = 2\pi,$$

we have then

$$\left.\begin{aligned} &a_1 \cos\beta_1 + a_2 \cos(\beta_1 + \beta_2) + a_3 \cos(\beta_1 + \beta_2 + \beta_3) + \ldots + a_n = 0 \\ &a_1 \sin\beta_1 + a_2 \sin(\beta_1 + \beta_2) + a_3 \sin(\beta_1 + \beta_2 + \beta_3) + \ldots \\ &\qquad\qquad + a_{n-1} \sin(\beta_1 + \beta_2 + \ldots + \beta_{n-1}) = 0 \end{aligned}\right\} \ldots (10),$$

the two fundamental relations between the sides and angles of a polygon. If there are only three sides, these relations reduce to (1) and (2) respectively, remembering that $\beta_1 = \pi - A_2$, $\beta_2 = \pi - A_3$.

128. In the first equation in (10), take a_n over to the other side of the equation, then square both sides of each equation and add; in the result the coefficient of $2a_ra_s$ is

$$\cos(\beta_1 + \beta_2 + \ldots + \beta_r)\cos(\beta_1 + \beta_2 + \ldots + \beta_s)$$
$$+ \sin(\beta_1 + \beta_2 + \ldots + \beta_r)\sin(\beta_1 + \beta_2 + \ldots + \beta_s),$$

or

$$\cos(\beta_{r+1} + \beta_{r+2} + \ldots + \beta_s);$$

this is the cosine of the angle θ_{rs} between the positive directions of the sides a_r and a_s; we thus obtain the formula

$$a_n^2 = a_1^2 + a_2^2 + \ldots + a_{n-1}^2 + 2a_1a_2\cos\theta_{12} + \ldots + 2a_ra_s\cos\theta_{rs} + \ldots (11),$$

which is analogous to the formulae (3), to which it reduces when $n = 3$. In the formula (11), r and s are each less than n and are unequal.

The area of a polygon.

129. The area of a polygon is given by the expression

$$\tfrac{1}{2}(a_1a_2\sin\theta_{12} + \ldots + a_ra_s\sin\theta_{rs} + \ldots) \ \ldots\ldots\ (12),$$

or $\frac{1}{2}\Sigma a_ra_s\sin\theta_{rs}$, the summation being taken for all different values of r and s; if we suppose s is always the greater of the two quantities r and s, the angle θ_{rs} is, as in the last Article, the sum of the external angles $\beta_{r+1} + \beta_{r+2} + \ldots + \beta_s$. To prove this formula, we shall first shew that in the case of a triangle it reduces to the expression $\frac{1}{2}a_2a_3\sin A_1$, and shall then shew that if it holds for a polygon of $n-1$ sides, it also holds for one of n sides

We have in the case of the triangle $A_1A_2A_3$, in which $A_1A_2 = a_1$,

$$\theta_{12} = \pi - A_2,\ \ \theta_{23} = \pi - A_3,\ \ \theta_{13} = 2\pi - A_2 - A_3,$$

hence in this case $\frac{1}{2}\Sigma a_r a_s \sin\theta_{rs}$ is equal to

$$\tfrac{1}{2}(a_1a_2 \sin A_2 + a_2a_3 \sin A_3 - a_1a_3 \sin A_1) \text{ or } \tfrac{1}{2}a_2a_3 \sin A_1,$$

thus the formula holds when $n = 3$.

Now suppose the formula true for a polygon of sides

$$a_1,\ a_2,\ \ldots\ldots\ a'_{n-1},$$

so that the area of the polygon is

$$\tfrac{1}{2}\Sigma a_r a_s \sin\theta_{rs} + \tfrac{1}{2}a'_{n-1}\Sigma a_r \sin\theta_{n-1,r},$$

where r and s are each less than $n-1$. Now replace the side a'_{n-1} by two sides a_{n-1}, a_n, thus making a polygon of n sides; we have to add $\frac{1}{2}a_{n-1}a_n \sin\theta_{n-1,n}$; the area of the polygon of n sides is then

$$\tfrac{1}{2}\Sigma a_r a_s \sin\theta_{rs} + \tfrac{1}{2}a'_{n-1}\Sigma a_r \sin\theta'_{n-1,r} + \tfrac{1}{2}a_{n-1}a_n \sin\theta_{n-1,n}.$$

Now we have, by projecting the side a'_{n-1} on a_r,

$$a'_{n-1}\sin\theta'_{r,n-1} = a_{n-1}\sin\theta_{r,n-1} + a_n \sin\theta_{r,n},$$

hence the above expression becomes

$$\tfrac{1}{2}\Sigma a_r a_s \sin\theta_{rs} + \tfrac{1}{2}\Sigma a_r(a_{n-1}\sin\theta_{r,n-1} + a_n \sin\theta_{r,n}) + \tfrac{1}{2}a_{n-1}a_n \sin\theta_{n-1,n},$$

or

$$\tfrac{1}{2}\Sigma a_r a_s \sin\theta_{rs},$$

where r and s have all different values from 1 up to n, such that $r < s$.

The formula (12) has been shewn to be true when $n = 3$, and is therefore true for $n = 4$, &c., and therefore holds generally.

It should be observed that in the formula (12) the coefficient of a_1 vanishes, in virtue of the second equation in (10); the formula therefore becomes $\frac{1}{2}\Sigma a_r a_s \sin\theta_{r,s}$, where r and s have all values from 2 up to n, s being always greater than r.

EXAMPLES ON CHAPTER X.

Prove the following relations in Examples 1—11, for a triangle ABC.

1. $a \sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

2. $a^3 \cos A+b^3\cos B+c^3\cos C=abc(1+4\cos A\cos B\cos C).$

3. $a^2\cos C+c^2\cos A=\dfrac{c+a}{2b}\{b^2+(c-a)^2\}.$

4. $a\cos A\cos 2A+b\cos B\cos 2B+c\cos C\cos 2C$
$$+4\cos A\cos B\cos C(a\cos A+b\cos B+c\cos C)=0.$$

5. $a^2\cos 2(B-C)=b^2\cos 2B+c^2\cos 2C+2bc\cos(B-C).$

6. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc.$

7. $c^3=a^3\cos 3B+3a^2b\cos(2B-A)+3ab^2\cos(B-2A)+b^3\cos 3A.$

8. $(\cot\frac{1}{2}A-\tan\frac{1}{2}B-\tan\frac{1}{2}C)^{\frac{1}{2}}+(\cot\frac{1}{2}B-\tan\frac{1}{2}C-\tan\frac{1}{2}A)^{\frac{1}{2}}$
$$+(\cot\tfrac{1}{2}C-\tan\tfrac{1}{2}A-\tan\tfrac{1}{2}B)^{\frac{1}{2}}=(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C)^{\frac{1}{2}}.$$

9. $b^2+c^2-2bc\cos(A+60^\circ)=c^2+a^2-2ca\cos(B+60^\circ)$
$$=a^2+b^2-2ab\cos(C+60^\circ);$$

interpret this result geometrically.

10 $\cos\frac{1}{2}B\sin(\frac{1}{2}B+C):\cos\frac{1}{2}C\sin(\frac{1}{2}C+B)::a+c:a+b.$

11. $(a+b)\sin B=2b\sin(B+\frac{1}{2}C)\cos\frac{1}{2}C.$

12. Prove that, if the sides of a triangle be in A.P., the cotangents of its semi-angles are in A.P.

13. If the squares of the sides of a triangle are in A.P., shew that the tangents of its angles are in H.P.

14. If $1-\cos A$, $1-\cos B$, $1-\cos C$ are in H.P., shew that $\sin A$, $\sin B$, $\sin C$ are in H.P.

15. If $b-a=mc$, prove that $A=\cos^{-1}(m\cos\frac{1}{2}C)-\frac{1}{2}C$,

and
$$\cot\tfrac{1}{2}(B-A)=\frac{1+m\cos B}{m\sin B}.$$

16. Prove that, in a triangle, $\cos A+\cos B+\cos C>1$ and $\not>\frac{3}{2}$.

17. Prove that, in a triangle, $\tan^2\frac{1}{2}B\tan^2\frac{1}{2}C+\tan^2\frac{1}{2}C\tan^2\frac{1}{2}A+\tan^2\frac{1}{2}A\tan^2\frac{1}{2}B<1$, and that if one angle approaches indefinitely near to two right angles, the least value of the expression is $\frac{1}{2}$.

18 Prove that a triangle is equilateral if $\cot A+\cot B+\cot C=\sqrt{3}$

19. If in a triangle,

$$\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C + 4 \cot A \cot B \cot C$$
$$= \sec \tfrac{1}{2}A \sec \tfrac{1}{2}B \sec \tfrac{1}{2}C + 4 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C,$$

prove that one angle is 60°.

20. If in a triangle, $\cos A = \cos B \cos C$, prove that $\cot B \cot C = \frac{1}{2}$.

21. If θ be an angle determined from $\cos\theta = \dfrac{a-b}{c}$, prove that

$$\cos \tfrac{1}{2}(A-B) = \frac{(a+b)\sin\theta}{2\sqrt{ab}}, \quad \text{and} \quad \cos \tfrac{1}{2}(A+B) = \frac{c\sin\theta}{2\sqrt{ab}}.$$

22. If O is a point inside an equilateral triangle, prove that

$$\cos(BOC - 60^\circ) = \frac{BO^2 + CO^2 - AO^2}{2BO\,.\,CO}.$$

23. If $c = b + \frac{1}{2}a$, and BC is divided in O so that $BO : OC :: 1 : 3$, prove that $\angle ACO = 2\angle AOC$.

24. If CD, CE make equal angles α with the base of a triangle ABC, shew that area ABC : area $CED :: c : 2b \sin A \cot \alpha$.

25. If AB be divided in C, D, so that $AC = CD = DB$, and if P be any other point, prove that $\sin APD \sin BPC = 4 \sin APC \sin BPD$.

26. If the sides of a parallelogram be a, b, and the angle between them be ω, prove that the product of the diagonals is $\{(a^2+b^2)^2 - 4a^2b^2\cos^2\omega\}^{\frac{1}{2}}$.

27. If D is the middle point of the side BC of a triangle, and $\angle BAD = \theta$, $\angle CAD = \phi$, shew that $\cot\theta - \cot\phi = \cot B - \cot C$.

28. A straight line divides the angle C of a triangle into segments α, β, and the side c into segments x, y, and is inclined to this side at an angle θ, prove that $x \cot\alpha - y\cot\beta = y \cot A - x \cot B = (x+y)\cot\theta$.

29. If the sides of a triangle are in A. P., and if the greatest angle exceeds the least by 90°, prove that the sides are as $\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$.

30. Prove geometrically, that in any triangle

$$a\cos\theta = b\cos(C-\theta) + c\cos(B+\theta), \ \theta \text{ being any angle.}$$

If a, b, c denote the sides AB, BC, CD of any plane quadrilateral, shew that

$$\frac{a\sin A - b\sin(A-B) + c\sin(A-B-C)}{a\cos A - b\cos(A-B) + c\cos(A-B-C)} = \tan 2A.$$

31. If a triangle ABC be such that it is possible to draw a straight line AD meeting BC in D, so that $\angle BAD$ is one-third of $\angle BAC$, and also BD is one-third of BC, prove that $a^2b^2 = (b^2-c^2)(b^2+8c^2)$.

32. BC is a side of a square; on the perpendicular bisector of BC, two points P, Q are taken, equidistant from the centre of the square; BP, CQ are joined and cut in A; prove that in the triangle ABC,

$$\tan A(\tan B - \tan C)^2 + 8 = 0.$$

33. If
$$\left.\begin{array}{l} y^2+z^2-2yz\cos\alpha=a^2 \\ z^2+x^2-2zx\cos\beta=b^2 \\ x^2+y^2-2xy\cos\gamma=c^2 \end{array}\right\} \text{ and } \alpha+\beta+\gamma=2\pi,$$
prove that
$$(yz\sin\alpha+zx\sin\beta+xy\sin\gamma)^2=\tfrac{1}{4}(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4).$$

34. If A, B, C are angles of a triangle, and x, y, z are real quantities satisfying the equation
$$\frac{y\sin C-z\sin B}{x-y\cos C-z\cos B}=\frac{z\sin A-x\sin C}{y-z\cos A-x\cos C},$$
then will
$$\frac{x}{\sin A}=\frac{y}{\sin B}=\frac{z}{\sin C}.$$

35. Prove that the area of the greatest rectangle that can be inscribed in a sector of a circle of radius R is $R^2\tan\frac{1}{2}\alpha$, where 2α is the angle of the sector.

36. Shew how to construct the right-angled triangle of minimum area which has its vertices on three given parallel straight lines; and if a, b are the distances of the middle line from the other two, shew that the hypothenuse makes with the parallel lines an angle $\cot^{-1}\dfrac{a-b}{a+b}$.

37. If the angles of a triangle computed from slightly erroneous measurements of the lengths of the sides be A, B, C, prove that if α, β, γ be the approximate errors of lengths, the consequent errors of the cotangents of the angles are proportional to
$$\operatorname{cosec} A\,(\beta\cos C+\gamma\cos B-\alpha),\quad \operatorname{cosec} B\,(\gamma\cos A+\alpha\cos C-\beta),$$
$$\operatorname{cosec} C\,(\alpha\cos B+\beta\cos A-\gamma).$$

38. Prove that, if in measuring the three sides of a triangle, small errors x, y be made in two of them a, b, the error in the angle C is
$$-\left(\frac{x}{a}\cot B+\frac{y}{b}\cot A\right),$$
and find the errors in the other angles.

39. The area of a triangle is determined by measuring the lengths of the sides, and the limit of error possible either in excess or defect in measuring any length is n times the length, where n is a small quantity. Prove that in the case of a triangle of sides 110, 81, 59, the limit of error possible in its area is about $3{\cdot}1433\,n$ times the area.

40. Prove that the cosines c_1, c_2, c_3, c_4 of the four angles of a quadrilateral satisfy the relation
$$\begin{aligned}(c_1^4+c_2^4+c_3^4+c_4^4)&-2(c_1^2c_2^2+c_2^2c_3^2+c_3^2c_1^2+c_4^2c_1^2+c_4^2c_2^2+c_4^2c_3^2)\\ &+4(c_2^2c_3^2c_4^2+c_3^2c_4^2c_1^2+c_4^2c_1^2c_2^2+c_1^2c_2^2c_3^2)\\ &+4c_1c_2c_3c_4(2-c_1^2-c_2^2-c_3^2-c_4^2)=0.\end{aligned}$$

CHAPTER XI.

THE SOLUTION OF TRIANGLES.

130. We shall now proceed to apply the formulae obtained in the preceding Chapter to the solution of triangles, that is, when the magnitudes of three of the six parts are given, to find the magnitude of the remaining three parts; one at least of the three given parts must be a side. We shall generally select such formulae as can be used for numerical computation by means of logarithms, as these formulae only are of use in practice.

The solution of triangles is made to depend upon a knowledge of the numerical values of circular functions of the angles, hence since such circular functions are the ratios of the sides of right-angled triangles, it is seen that the solution of all triangles is really performed by dividing up the triangles into right-angled ones.

The solution of right-angled triangles.

131. Suppose the angle C of a triangle to be 90°, then this is one of the given parts, and we can solve the triangle in the various cases in which there are two other parts given, one at least being a side.

(1) Suppose the two sides a, b to be given; then the angle A can be determined from the formula $\tan A = a/b$, and B is then found as the complement of A; also $c = a \operatorname{cosec} A$, which determines c, when A has been found; the logarithmic formulae for solving the triangle are then

$$L \tan A = 10 + \log a - \log b,$$

$$B = 90^\circ - A,$$

$$\log c = \log a - L \sin A + 10.$$

(2) Suppose the hypothenuse c and one side a to be given, then the angle A is determined by means of the formula $\sin A = a/c$, B is found as the complement of A, and b is found from the formula $b = c \cos A$, or from $b^2 = c^2 - a^2$.

The logarithmic formulae are

$$L \sin A = 10 + \log a - \log c,$$

$$B = 90° - A,$$

and $$\log b = \log c + L \cos A - 10$$

or $$\log b = \tfrac{1}{2} \log (c + a) + \tfrac{1}{2} \log (c - a).$$

(3) Suppose the hypothenuse c and one angle A are given, then B is found at once as the complement of A; a is found from $a = c \sin A$, and b as in the last case.

The formulae are

$$\log a = \log c + L \sin A - 10,$$

$$B = 90° - A,$$

$$\log b = \log c + L \cos A - 10$$

or $$\log b = \tfrac{1}{2} \log (c + a) + \tfrac{1}{2} \log (c - a).$$

(4) Suppose one side a and one angle A to be given, then B is $90° - A$, c is $a \operatorname{cosec} A$, and b is found as in the last two cases; the formulae are

$$\log c = \log a - L \sin A + 10,$$

$$B = 90° - A,$$

$$\log b = \log c + L \cos A - 10$$

or $$\log b = \tfrac{1}{2} \log (c + a) + \tfrac{1}{2} \log (c - a).$$

132. In certain cases, the formulae of the last Article are inconvenient, for example in case (2) if the angle A is nearly $90°$, it cannot be conveniently determined from the equation $\sin A = a/c$, since the differences for consecutive sines are in this case insensible, we therefore use another formula; from the theorem (4) of Chap. X. we obtain $b \tan \tfrac{1}{2} B = c - a$, $b \cot \tfrac{1}{2} B = c + a$, hence $\tan^2 \tfrac{1}{2} B = \dfrac{c - a}{c + a}$, thus we have $\tan (45° - \tfrac{1}{2} A) = \left(\dfrac{c - a}{c + a}\right)^{\frac{1}{2}}$, and this formula, being free from the objection, may be used to determine A.

Again, in cases (3) and (4), the formula $b = c \cos A$ is inconvenient if A is very small; we may then use the formula $b = c - c \sin A \tan \tfrac{1}{2} A$.

133. Various approximate formulae may be found for the solution of right-angled triangles. Let us denote by α, β the circular measures of the angles A, B respectively.

(1) An approximate form of the formula $a=c\cos B$ is

$$a=c\left(1-\tfrac{1}{2}\beta^2+\tfrac{1}{24}\beta^4\right),$$

which is obtained by taking the first three terms of the expansion of $\cos B$ in powers of the circular measure of B; this formula may then be used for approximate calculation of a, when c and B are given, provided β is not too large.

(2) Since $\sin A=a/c$, we have $\alpha-\frac{1}{6}\alpha^3+\frac{1}{120}\alpha^5=a/c$, approximately; to obtain α in terms of a/c, we have as a first approximation $\alpha=a/c$, and as a second approximation $\alpha=\dfrac{a}{c}+\dfrac{1}{6}\left(\dfrac{a}{c}\right)^3$; the third approximation is

$$\alpha=\frac{a}{c}+\frac{1}{6}\left\{\frac{a}{c}+\frac{1}{6}\left(\frac{a}{c}\right)^3\right\}^3-\frac{1}{120}\left(\frac{a}{c}\right)^5$$

or

$$\alpha=\frac{a}{c}+\frac{1}{6}\left(\frac{a}{c}\right)^3+\frac{3}{40}\left(\frac{a}{c}\right)^5,$$

which may be used to calculate α.

(3) From the equation $\tan\frac{1}{2}B=\left(\dfrac{c-a}{c+a}\right)^{\frac{1}{2}}$ we can obtain the approximate formula $\frac{1}{2}\beta=\left(\dfrac{c-a}{c+a}\right)^{\frac{1}{2}}\left\{1-\dfrac{1}{3}\left(\dfrac{c-a}{c+a}\right)+\dfrac{1}{5}\left(\dfrac{c-a}{c+a}\right)^2\right\}$.

(4) Using Snellius' formula $\phi=\dfrac{3\sin 2\phi}{2(2+\cos 2\phi)}$, for the circular measure of an angle (see Ex. 32, p. 138), in which the approximate error is $\frac{4}{45}\phi^5$, put $2\phi=\beta$, we then obtain the formula $\beta=\dfrac{3b}{2c+a}$, and the error is approximately $\frac{1}{180}\beta^5$; thus B is given in degrees by the approximate equation

$$B=\frac{3b}{2c+a}\times 57^\circ{\cdot}2957.$$

The solution of oblique-angled triangles.

134. *To solve a triangle when the three sides are given;* any one of the formulae

$$\sin\tfrac{1}{2}A=\left\{\frac{(s-b)(s-c)}{bc}\right\}^{\frac{1}{2}},\quad \cos\tfrac{1}{2}A=\left\{\frac{s(s-a)}{bc}\right\}^{\frac{1}{2}},$$

$$\tan\tfrac{1}{2}A=\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}^{\frac{1}{2}},$$

with the corresponding formulae for the other angles, may be used; these formulae are adapted for logarithmic calculation.

EXAMPLE.

The sides of a triangle are proportional to 4, 7, 9; *find the angles, having given*

$$\log 2 = {\cdot}301030,$$

$$L \tan 12^\circ\, 36' = 9{\cdot}349329, \ \textit{diff. for } 1' = {\cdot}000593,$$

$$L \tan 24^\circ\ \ 5' = 9{\cdot}650281, \ \textit{diff. for } 1' = {\cdot}000339.$$

We find $s = 10$, $s - a = 6$, $s - b = 3$, $s - c = 1$, and hence $\tan \frac{1}{2} A = \sqrt{1/20}$, $\tan \frac{1}{2} B = \sqrt{2/10}$, thus $L \tan \frac{1}{2} A = 10 - \frac{1}{2}(1 + {\cdot}301030) = 9\ 349485$

and $$L \tan \tfrac{1}{2} B = 10 + \tfrac{1}{2}({\cdot}301030 - 1) = 9{\cdot}650515.$$

To find A, we have $9{\cdot}349485 - 9{\cdot}349329 = {\cdot}000156$, and $\frac{156}{593} \,.\, 60'' = 15''\ 8$ approximately, hence $\frac{1}{2} A = 12^\circ\, 36'\, 15''{\cdot}8$, or $A = 25^\circ\, 12'\, 31''{\cdot}6$.

To find B, we have $9{\cdot}650515 - 9{\cdot}650281 = {\cdot}000234$ and $\frac{234}{339} \,.\, 60'' = 41''{\cdot}4$ approximately, hence $\frac{1}{2} B = 24^\circ\, 5'\, 41''{\cdot}4$, or $B = 48^\circ\, 11'\, 22''{\cdot}8$; also

$$C = 180^\circ - A - B = 106^\circ\, 36'\, 5''{\cdot}6;$$

thus we have found the approximate values of the angles.

135. *To solve a triangle when two sides and the included angle are given.*

Suppose b, c, and A are the given parts, then B and C may be determined from the formula

$$\tan \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \tfrac{1}{2} A,$$

together with $B + C = 180^\circ - A$; the logarithmic formula is

$$L \tan \tfrac{1}{2}(B - C) = \log(b - c) - \log(b + c) + L \cot \tfrac{1}{2} A.$$

Having found B and C, the side a may be found from any one of the three formulae

$$\log a = \log c + L \sin A - L \sin C,$$

$$\log a + L \cos \tfrac{1}{2}(B - C) = \log(b + c) + L \sin \tfrac{1}{2} A,$$

$$\log a + L \sin \tfrac{1}{2}(B - C) = \log(b - c) + L \cos \tfrac{1}{2} A.$$

We may also determine a thus:—Since $a^2 = b^2 + c^2 - 2bc \cos A$ we have

$$a^2 = (b + c)^2 - 4bc \cos^2 \tfrac{1}{2} A,$$

hence $a = (b + c) \cos \phi$, where ϕ is given by

$$\sin \phi = \frac{2\sqrt{bc} \cos \frac{1}{2} A}{b + c};$$

thus we may first find ϕ by the logarithmic formulae

$$L\sin\phi = \log 2 + \tfrac{1}{2}\log b + \tfrac{1}{2}\log c + L\cos\tfrac{1}{2}A - \log(b+c),$$

and then determine a by the formula

$$\log a = \log(b+c) + L\cos\phi - 10.$$

EXAMPLE.

If a = 123, c = 321, B = 29° 16′, *find* A, C, b, *having given*

log $99 = 1{\cdot}9956352$, *L sin* $29° 16' = 9{\cdot}6891978$,

log $123 = 2{\cdot}0899051$, *L sin* $15° 42' = 9{\cdot}4323285$, *diff. for* $1'' = 74{\cdot}87$,

log $2220 = 3{\cdot}3463530$, *L cot* $14° 38' = 10{\cdot}5831901$,

log $2221 = 3{\cdot}3465486$, *L tan* $59° 39' = 10{\cdot}2324552$, *diff. for* $1'' = 48{\cdot}27$.

We have
$$\begin{aligned} L\tan\tfrac{1}{2}(C-A) &= L\cot 14° 38' + \log 99 - \log 222 \\ &= 10{\cdot}5831901 + 1{\cdot}9956352 - 2{\cdot}3463530 \\ &= 10{\cdot}2324723. \end{aligned}$$

Now $10{\cdot}2324723 - 10{\cdot}2324552 = {\cdot}0000171$, and $\dfrac{171}{48{\cdot}27} = 3{\cdot}5$ approximately, hence $\tfrac{1}{2}(C-A) = 59° 39' 3''{\cdot}5$, also $\tfrac{1}{2}(C+A) = 75° 22'$, therefore $A = 15° 42' 56''{\cdot}5$, $C = 135° 1' 3''{\cdot}5$.

Again $\log b = 9{\cdot}6891978 + 2{\cdot}0899051 - L\sin 15° 42' 56''{\cdot}5$,

and $56{\cdot}5 \times 74{\cdot}87 = 4230{\cdot}155$, hence $L\sin 15° 42' 56''{\cdot}5 = 9{\cdot}4327515$,

therefore $\log b = 2{\cdot}3463514$, so that $b = 222 - \frac{16}{1956} = 221{\cdot}992$.

136. *To solve a triangle when two sides and the angle opposite one of them are given.*

This is usually known as the ambiguous case.

Suppose a, c, and A are the given parts, then $\sin C$ is determined from the equation $\sin C = \dfrac{c}{a}\sin A$; when $\sin C$ is thus found, there are in general, if $c\sin A \not> a$, two values of C less than 180°, the one acute and the other obtuse, whose sine has the value determined; we must consider three different cases:—

(1) if $c\sin A > a$, we have $\sin C > 1$, which is impossible, and indicates that there is no triangle with the given parts;

(2) if $c\sin A = a$, then $\sin C = 1$, and the only value of C is 90°. If $A < 90°$, there is one triangle with the given parts, and that one a right-angled triangle. If $A > 90°$, the value $C = 90°$ is inadmissible, and there exists no triangle with the given parts.

(3) if $c \sin A < a$, then $\sin C < 1$, and there are two values of C, one acute, the other obtuse

(α) if $c < a$, we must have $C < A$, hence C must be acute, thus there is only one triangle with the given parts;

(β) if $c > a$, the angle C is not restricted to being acute, and both values are admissible, provided $A < 90°$; but if $A > 90°$ neither value is admissible since $C > A$. There are two triangles or none with the given parts according as $A < 90°$ or $A > 90°$;

(γ) if $c = a$, then $C = A$ or $180° - A$; for the latter value of C two sides of the triangle are coincident, the first then gives the only value of C for which there is a triangle of finite area, but this is only admissible when $A < 90°$.

We may state the above results thus:

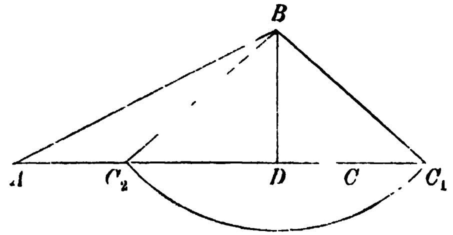
$$\begin{array}{ll}
c \sin A > a, & \text{no solution} \\
c \sin A = a, & A < 90°, \text{ one solution} \\
c \sin A = a, & A > 90°, \text{ no solution} \\
c \sin A < a, & \begin{cases} c < a, \text{ one solution} \\ c = a, \; A < 90°, \text{ one solution} \\ c = a, \; A > 90°, \text{ no solution} \\ c > a, \; A < 90°, \text{ two solutions} \\ c > a, \; A > 90°, \text{ no solution} \end{cases}
\end{array}$$

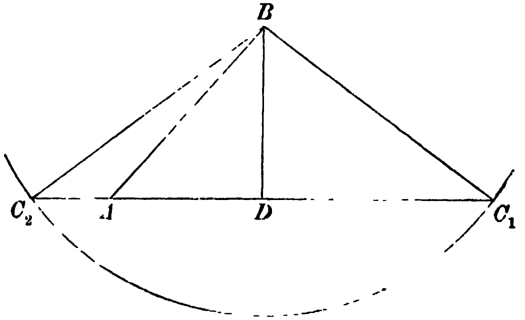
When C is nearly $90°$, it cannot be conveniently determined by means of its sine; in that case we may use one of the formulae

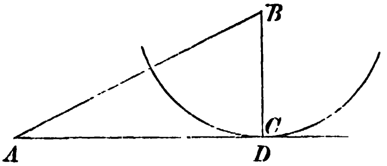
$$\tan C = \pm \frac{c \sin A}{\sqrt{(a + c \sin A)(a - c \sin A)}}, \quad \tan(45° + \tfrac{1}{2}C) = \pm \sqrt{\frac{a + c \sin A}{a - c \sin A}}.$$

137. It is instructive to investigate geometrically the different cases considered in the last Article.

From B draw BD perpendicular to the side b, then $BD = c \sin A$; with centre B and radius a, describe a circle; then if a is less than $c \sin A$, this circle will not cut the side AC and no triangle with the given parts can be drawn, but if $a > c \sin A$, the circle will cut AC in two points, C_1 and C_2. In the case $a < c$ and $A < 90°$, both C_1 and C_2 are, as in Fig. (1), on the same side of A, and the two triangles ABC_1 and ABC_2 have each the given parts, the angles AC_1B, AC_2B being supplementary. When $a < c$, and $A > 90°$, A will be beyond C_1, and no triangle with the given parts exists. If $a > c$, then C_1 and C_2 are on







opposite sides of A, and only the triangle ABC_1 has the given parts. The triangle ABC_2, in this latter case, has the angle at A not equal to A, but to $180° - A$, and therefore does not satisfy the given conditions.

If $a = c \sin A$, the circle touches AC at D, and the right-angled triangle ADB is the one triangle with the given parts, provided $A < 90°$.

We remark that since, in Fig. (1),

$$AD = c \cos A, \quad \text{and} \quad C_1D = C_2D = \sqrt{a^2 - c^2 \sin^2 A},$$

the two values of b are

$$c \cos A + \sqrt{a^2 - c^2 \sin^2 A} \quad \text{and} \quad c \cos A - \sqrt{a^2 - c^2 \sin^2 A},$$

these values being both positive when there are two solutions; we may also obtain these values of b as the roots of the quadratic equation in b,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

138. *To solve a triangle when one side and two angles are given,*

Suppose a the given side, and A, C the given angles, then B is determined from the equation $B = 180° - A - C$, and the sides b, c will be determined by means of the formulae

$$\log b = \log a + L\sin B - L\sin A,$$
$$\log c = \log a + L\sin C - L\sin A.$$

EXAMPLE.

If $a = 10$, $A = 51° \, 30' \, 40''$, $B = 76°$, *find* b, *having given*

$$\log 12396 = 4{\cdot}0932816, \quad L\sin 76° = 9{\cdot}9869041,$$
$$\log 12397 = 4{\cdot}0933166, \quad L\sin 51° \, 30' = 9{\cdot}8935444,$$
$$L\sin 51° \, 31' = 9{\cdot}8936448.$$

We have $\log b = 9{\cdot}9869041 + 1 - L\sin 51° \, 30' \, 40''$

and $L\sin 51° \, 30' \, 40'' = 9{\cdot}8935444 + \frac{40}{60} \times {\cdot}0001004$
$= 9{\cdot}8936113,$

hence $\log b = 1{\cdot}0932928$, therefore $b = 12\,396 + \frac{112}{350} \times {\cdot}001$,

or $b = 12{\cdot}3963$ approximately.

139. The expression $c\cos A \pm \sqrt{a^2 - c^2\sin^2 A}$ for b may be adapted to logarithmic calculation; let $\sin\phi = \frac{c}{a}\sin A$, then $b = \frac{a\sin(\phi \pm A)}{\sin A}$, thus ϕ having been determined from the equation $L\sin\phi = L\sin A + \log c - \log a$, we can determine b from $\log b = \log a + L\sin(\phi \pm A) - L\sin A$.

Denoting by α, β, γ the circular measures of the angles A, B, C, respectively, and by α', β', γ' the complements of α, β, γ, we obtain the following approximate formulae for the solution of triangles.

(1) Suppose A, C, a are given, C not being large; then from the formula

$$c = \frac{a\sin C}{\sin A}, \text{ we get the approximate formula}$$

$$c = a\operatorname{cosec} A\,\{\gamma - \tfrac{1}{6}\gamma^3 + \tfrac{1}{120}\gamma^5\}.$$

Also if A and C are both not large, we have

$$c = \frac{a\,(\gamma - \frac{1}{6}\gamma^3 + \frac{1}{120}\gamma^5 - \dots)}{\alpha - \frac{1}{6}\alpha^3 + \frac{1}{120}\alpha^5 - \dots},$$

hence c is given approximately by

$$c = a\frac{\gamma}{\alpha}\{1 + \tfrac{1}{6}(\alpha^2 - \gamma^2)\},$$

which may be used for calculating c.

(2) Suppose, as in the last case, that A, C, a are given; also suppose C is nearly 90°, then $c = \frac{a\cos\gamma'}{\sin A}$, therefore $c = \frac{a}{\sin A}(1 - \frac{1}{2}\gamma'^2 + \frac{1}{24}\gamma'^4)$ may be used to determine c approximately.

If both A and C are nearly 90°, we have

$$c=\frac{a\cos\gamma'}{\cos\alpha'},\quad\text{or}\quad c=\frac{a\left(1-\frac{1}{2}\gamma'^2+\ldots\right)}{1-\frac{1}{2}\alpha'^2+\ldots},$$

therefore $$c=a\left\{1-\tfrac{1}{2}\left(\gamma'^2-\alpha'^2\right)\right\}$$

gives c approximately.

140. We shall give a few examples of the solution of triangles, when instead of sides and angles there are other data.

(1) Suppose the three perpendiculars from the angles on the opposite sides given; denote them by p_1, p_2, p_3, we have then $ap_1=bp_2=cp_3=2$ area of triangle. Now since

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}$$

we have $$\cos\tfrac{1}{2}A=\sqrt{\frac{(p_2p_3+p_3p_1+p_1p_2)(-p_2p_3+p_3p_1+p_1p_2)}{4p_1{}^2p_2p_3}},$$

which determines A; also $p_2=c\sin A$, hence c is determined when A is known.

(2) Suppose the perimeter and the angles of the triangle given. We have

$$s=R(\sin A+\sin B+\sin C),$$

hence R is determined, and the sides are then

$$2R\sin A,\quad 2R\sin B,\quad 2R\sin C,\quad\text{or}\quad a=\frac{2s\sin A}{\sin A+\sin B+\sin C},$$

with similar values for b and c; this value of a reduces to $\dfrac{s\sin\frac{1}{2}A}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$, which is adapted to logarithmic calculation.

(3) Suppose the base, height, and difference of the angles at the base given. Let a be the base, p the height and $B-C=2\alpha$ the given difference: then since $B+C=180°-A$, we have $B=90°+\alpha-\frac{1}{2}A$, $C=90°-\alpha-\frac{1}{2}A$, also

$$a=p(\cot B+\cot C)=p\left\{\tan\left(\tfrac{1}{2}A-\alpha\right)+\tan\left(\tfrac{1}{2}A+\alpha\right)\right\},$$

therefore $\dfrac{a}{p}=\dfrac{2\sin A}{\cos A+\cos 2\alpha}$, hence $\cos A$ is given by the quadratic

$$a^2(\cos A+\cos 2\alpha)^2=4p^2(1-\cos^2 A)$$

or $$\cos^2 A\,(a^2+4p^2)+2a^2\cos 2\alpha\,.\cos A=4p^2-a^2\cos^2 2\alpha,$$

the solution of which is

$$\cos A=-\frac{a^2\cos 2\alpha}{a^2+4p^2}\pm\frac{2p\left(4p^2+a^2\sin^2 2\alpha\right)^{\frac{1}{2}}}{a^2+4p^2};$$

these are two values of $\cos A$ corresponding to two solutions of the problem.

Solve the triangle with the following data:

(4) C, c, a + b.

(5) B, a, b + c.

(6) *The area and the angles.*

(7) C, c + a, c + b.

(8) *The angles and the height.*

The solution of polygons.

141. The relations between the sides and angles of polygons, and the methods of solving a polygon when a certain number of sides and angles are given, have been considered by Carnot[1], L'Huilier[2], Lexell[3], and others. The two fundamental equations in this so-called Polygonometry have been given in Art. 127.

In order that a polygon of n sides may be determinate, $2n-3$ of its $2n$ parts must be given, and of these at least $n-2$ must be sides. To prove this, suppose the polygon divided, by means of a diagonal, into a triangle and a polygon of $n-1$ sides; if the sides and angles of the latter polygon were determined, we should only require to know two parts of the triangle in order to determine the figure completely, since one side of the triangle is already determined as a side of the polygon, hence to determine a polygon of n sides we require to know two more parts than for a polygon of $n-1$ sides; since therefore for a triangle three parts must be given, one of which is a side, for a polygon of n sides we must have $3+2(n-3)$, that is $2n-3$ parts given. If of these $2n-3$ parts, only $n-3$ were sides, we should have n angles given; but if $n-1$ angles are given, the nth is also given, so that only $2n-4$ independent parts would be given, thus at least $n-2$ of the given parts must be sides.

In some cases, a polygon can be conveniently solved by dividing it by means of diagonals into triangles, taking the diagonals for parts to be determined; this method is however not always convenient, as may be seen, for example, by considering the case of a quadrilateral when two opposite sides and three angles are given.

142. *To solve a polygon of* n *sides, when* n − 1 *sides and* n − 2 *angles are given.*

(1) Suppose the angles to be found are adjacent to the side to be found. We shall, as in Art. 127, use the external angles $\beta_1, \beta_2 \dots \beta_n$ between the sides, instead of the internal angles;

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[1] Carnot, *Geometrie der Stellung.*

[2] L'Huilier, *Polygonométrie.* Geneva, 1789.

[3] Lexell, *Nov. Comm. Petrop.*, Vols. XIX. XX.

suppose a_n the side to be found, then from the second equation (10) of Art. 127, we have

$$\sin\beta_1\{a_1+a_2\cos\beta_2+a_3\cos(\beta_2+\beta_3)+\ldots+a_{n-1}\cos(\beta_2+\ldots+\beta_{n-1})\}$$
$$=-\cos\beta_1\{a_2\sin\beta_2+a_3\sin(\beta_2+\beta_3)+\ldots+a_{n-1}\sin(\beta_2+\ldots+\beta_{n-1})\},$$

hence

$$\tan\beta_1=-\frac{a_2\sin\beta_2+a_3\sin(\beta_2+\beta_3)+\ldots+a_{n-1}\sin(\beta_2+\ldots+\beta_{n-1})}{a_1+a_2\cos\beta_2+a_3\cos(\beta_2+\beta_3)+\ldots+a_{n-1}\cos(\beta_2+\ldots+\beta_{n-1})},$$

this determines β_1 in terms of the given angles $\beta_2, \beta_3 \ldots \beta_{n-1}$ and the given sides $a_2, a_3 \ldots a_{n-1}$; it should be noticed that this equation is found by projecting the sides on a perpendicular to the unknown side; the remaining angle β_n is then determined from the relation $\beta_1+\beta_2+\ldots+\beta_n=2\pi$.

Having found β_1 and β_n, we can determine a_n from the equation obtained by projecting the sides on a_n,

$$a_n=-\{a_1\cos\beta_1+a_2\cos(\beta_1+\beta_2)+\ldots\},$$

or by means of the equation (11) of Art. 128, which gives a_n^2 in terms of the squares and products of the other sides and of the cosines of the angles between the sides.

(2) Suppose the angles to be found are adjacent to one another but not to the side which is to be found. We shall take a_n as the side to be found, and β_r, β_{r+1} the angles to be found,

then $\beta_r+\beta_{r+1}=2\pi-(\beta_1+\beta_2+\ldots+\beta_{r-1}+\beta_{r+2}+\ldots+\beta_n)$,

thus $\beta_r+\beta_{r+1}$ is known; also from the second equation (10)

$$a_r\sin(\beta_1+\beta_2+\ldots+\beta_r)=-a_1\sin\beta_1-a_2\sin(\beta_1+\beta_2)-\ldots$$
$$-a_{r-1}\sin(\beta_1+\beta_2+\ldots+\beta_{r-1})-a_{r+1}\sin(\beta_1+\ldots+\beta_{r+1})-\ldots$$
$$-a_{n-1}\sin(\beta_1+\ldots+\beta_n),$$

hence $\beta_1+\beta_2+\ldots+\beta_r$ can be determined, and therefore β_r.

The side a_n is then determined as in the last case.

(3) In the case in which the two unknown angles are not adjacent to one another, let *H*, *K* be the angular points at which the angles are unknown; join *HK*, then the polygon is divided into two polygons, in one of which all the sides except one are known, and all the angles except the two which are adjacent to the unknown side. We can solve this polygon as in (1), determining *HK* and the angles *H* and *K*.

In the other polygon we now have all the sides except one given, and all the angles except two adjacent ones; this polygon can therefore be solved as in (2); we have then all the sides of the given polygon determined, and the angles at H and K are determined by adding the two parts into which they were divided by HK, and which have been separately found.

143. *To solve a polygon of* n *sides, when* n − 2 *sides and* n − 1 *angles are given.*

We determine the remaining angle at once from the condition

$$\beta_1 + \beta_2 + \dots + \beta_n = 2\pi.$$

To determine an unknown side a_r, use the equation

$$a_1 \sin \beta_1 + a_2 \sin (\beta_1 + \beta_2) + \dots + a_{n-1} \sin (\beta_1 + \beta_2 + \dots + \beta_{n-1}) = 0,$$

obtained by projecting perpendicularly to the other unknown side a_n. We can then determine a_n in a similar manner, or use the other fundamental equation.

144. *To solve a polygon of* n *sides, when the* n *sides and* n − 3 *angles are given.*

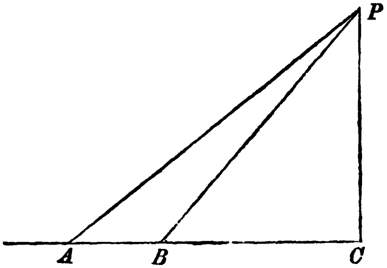
Let P, Q, R be the angular points at which the angles are not given; join PQ, QR, RP, then the polygon is divided into four parts, one of which is a triangle. In each of the parts except PQR, all the sides except one are given, and all the angles except those adjacent to those sides; we can therefore determine PQ, QR, RP, and the angles at P, Q, R. We can then find the angles of the triangle PQR, of which the sides have been determined. We obtain now by addition the angles at P, Q, R, of the given polygon.

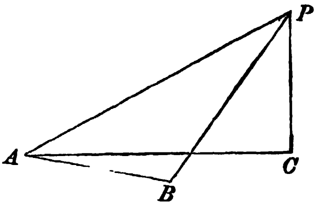
Heights and distances.

145. We shall now give some examples of the application of the solution of triangles to the determination of heights and distances. For fuller information on this subject, as for the description of instruments for measuring angles, we must refer to treatises on surveying. The angle which the distance from any point of observation to an object makes with the horizon is called the *elevation* or the *depression* of that object, according as the object is above or below the horizontal plane through the point of observation.

146. *To find the height of an inaccessible point above a horizontal plane.*

Let P be the inaccessible point and C its projection on the horizontal plane, let $PC = h$, and suppose any line $AB = a$, measured on the horizontal plane, if possible so that ABC is a straight line; let the elevations of P at A and B be measured, denote them by α and β; then $a = AC - BC = h(\cot\alpha - \cot\beta)$, therefore



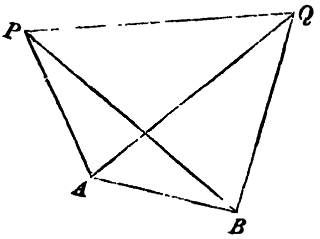


$$h = \frac{a \sin\alpha \sin\beta}{\sin(\beta - \alpha)}$$

which determines h. If it is impracticable to measure the base line directly towards C, let it be measured in any other direction; let the elevations α of P be measured at A, and also the angles $PAB = \gamma$, and $PBA = \delta$, then $PA = AB\dfrac{\sin\delta}{\sin(\gamma+\delta)}$, and $h = AP\sin\alpha$, therefore $h = a\dfrac{\sin\alpha\sin\delta}{\sin(\gamma+\delta)}$, thus h is determined.

147. *To find the distance between two inaccessible points.*

Let P and Q be the two objects, and let any base line $AB = a$ be measured, the points A, B being so chosen that P and Q are



both visible from each of them. At A measure the three angles $PAQ = \alpha$, $QAB = \beta$, $PAB = \gamma$; it should be observed that the angles PAQ, QAB are in general not in the same plane. At B measure the angles $PBA = \delta$, and $QBA = \epsilon$.

From the two triangles ABP, ABQ, we have

$$AP = a\frac{\sin\delta}{\sin(\gamma+\delta)},$$

and $AQ = a\dfrac{\sin\epsilon}{\sin(\beta+\epsilon)}$. Thus AP, AQ are determined by the formulae

$$\log AP = \log a + L\sin\delta - L\sin(\gamma+\delta),$$

$$\log AQ = \log a + L\sin\epsilon - L\sin(\beta+\epsilon).$$

In the triangle PAQ, we now know AP, AQ, and the angle $PAQ = \alpha$, we find then the angles APQ, AQP by means of the formulae

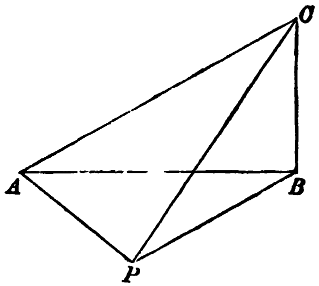
$$L\tan\tfrac{1}{2}(APQ - AQP) = L\cot\tfrac{1}{2}\alpha + \log(AQ - AP) - \log(AQ + AP),$$

$$APQ + AQP = 180° - \alpha.$$

We then find PQ by means of the formula

$$\log PQ = \log AP + L\sin\alpha - L\sin AQP.$$

148. *Pothenot's Problem.* To determine a point in the plane of a triangle at which the sides of the triangle subtend given angles.



Let α, β be the angles subtended by the sides AC, CB of a triangle ABC at the point P, and let x, y denote the angles PAC, PBC respectively; the position of P is found when the angles x and y are determined, for the distances PA and PB can be found by solving the triangles PAC, PBC.

We have $$x+y=2\pi-\alpha-\beta\mp C.$$

Also $$\frac{b\sin x}{\sin\alpha}=\frac{a\sin y}{\sin\beta}=PC.$$

Assume ϕ to be an auxiliary angle such that

$$\tan\phi=\frac{a\sin\alpha}{b\sin\beta},$$

therefore $\dfrac{\sin x}{\sin y}=\tan\phi$, hence $\dfrac{\sin x-\sin y}{\sin x+\sin y}=\tan(\phi-45^\circ)$,

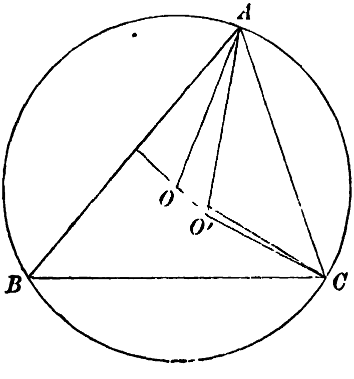
or $$\tan\tfrac{1}{2}(x-y)=\tan\tfrac{1}{2}(x+y)\tan(\phi-45^\circ)$$
$$=\tan(45^\circ-\phi)\tan\tfrac{1}{2}(\alpha+\beta+C);$$

thus $x-y$ can be found, and since $x+y$ is known, we can find x and y.

149. EXAMPLES.

(1) *It is observed that the elevation of the top of a mountain at each of the three angular points* A, B, C, *of a plane horizontal triangle* ABC, *is* a; *shew that the height is* $\frac{1}{2}$a *tan* a *cosec* A. *Shew also, that if there be a small error* n″ *in the elevation at* C, *the true height is very nearly* $\dfrac{1}{2}\dfrac{a\tan a}{\sin A}\left(1+\dfrac{\cos C}{\sin A\sin B}\cdot\dfrac{\sin n''}{\sin 2a}\right)$.

Let O be the projection of the top of the mountain on the plane ABC, we have then, if h is the height of the mountain, $h=OA\tan a=OB\tan a=OC\tan a$,



thus O is the centre of the circle round ABC, hence $OA=\frac{1}{2}a$ cosec A, or $h=\frac{1}{2}a\tan a$ cosec A. When the measurement of the elevation at C is $a+n''$, let O' be the projection of the top of the mountain, then since the elevations at A and B are equal, OO' is perpendicular to AB; let $h+x$ now be the height of the mountain. We find geometrically,

$$O'A=OA+OO'\cos C,\quad O'C=OC-OO'\cos(A-B),$$

when OO' is so small that its square may be neglected, hence

$$h+x=O'A\tan a=O'C\tan(a+n'')$$
$$=(OA+OO'\cos C)\tan a=\{OC-OO'\cos(A-B)\}\tan(a+n''),$$

hence $x=OO'.\cos C\tan a=-OO'\cos(A-B)\tan a+OC\sec^2 a.\sin n''$,

since $\tan(a+n'')=\tan a+\sec^2 a.\sin n''$, approximately; eliminating OO', we have

$$x\cos(A-B)\tan a=\cos C\tan a\,(OC\sec^2 a.\sin n''-x),$$

hence $$2x\sin A\sin B=OC\sec^2 a\cos C\sin n'',$$

therefore the true height $h+x$ is $\frac{1}{2}\frac{a\tan a}{\sin A}\left(1+\frac{\cos C}{\sin A\sin B}.\frac{\sin n''}{\sin 2a}\right)$

(2) *The sides of a triangle are observed to be* a$=5$, b$=4$, c$=6$, *but it is known that there is a small error in the measurement of* c; *examine which angle can be determined with the greatest accuracy.*

Let $6+x$ be the true value of the side c; let $A+\delta A$, $B+\delta B$, $C+\delta C$ be the angles of the triangle, the parts δA, δB, δC depending on x; we suppose x so small that its square may be neglected.

We have

$$\cos(A+\delta A)=\frac{16+(6+x)^2-25}{2.4(6+x)}=\frac{27+12x}{48(1+\frac{1}{6}x)}=\frac{27}{48}(1+\tfrac{12}{27}x-\tfrac{1}{6}x)=\frac{27}{48}(1+\tfrac{5}{18}x),$$

approximately, hence $\sin A.\delta A=-\frac{5}{32}x$.

Also $\cos(B+\delta B)=\frac{25+(6+x)^2-16}{2.5(6+x)}=\frac{3}{4}\left(1+\frac{x}{10}\right)$, hence $\sin B.\delta B=-\frac{3}{40}x$,

and $\cos(C+\delta C)=\frac{25+16-(6+x)^2}{2.5.4}=\frac{1}{8}\left(1-\frac{12x}{5}\right)$, hence $\sin C.\delta C=\frac{1}{10}x$.

Also $$\frac{\sin A}{5}=\frac{\sin B}{4}=\frac{\sin C}{6},$$

so that $$24.\delta A=40.\delta B=-15.\delta C.$$

Thus δB is numerically smaller than δA and δC, hence the angle B can be determined with the greatest accuracy.

EXAMPLES ON CHAPTER XI.

1. The sides of a triangle are 8, 7, 5; find the least angle, having given

$$\log 112=2{\cdot}0492180,$$
$$L\cos 19^\circ 6'=9{\cdot}9754083,\quad \text{diff. for } 60''={\cdot}0000437.$$

2. If in a triangle $a=65$, $b=16$, $C=60^\circ$, find the other angles, having given

$$\log 3={\cdot}4771213,\quad L\tan 46^\circ 20'=10{\cdot}0202203,$$
$$\log 7={\cdot}8450980,\quad L\tan 46^\circ 21'=10{\cdot}0204731.$$

3. The sides of a triangle are 3, 5, 7 feet; find the angles, having given

$$\log 13{\cdot}5 = 1{\cdot}1303338, \qquad \log 14 = 1{\cdot}1461280,$$
$$L\cos 10^\circ\, 53' = 9{\cdot}9921175, \qquad L\cos 10^\circ\, 54' = 9{\cdot}9920932.$$

4. If $B = 45^\circ$, $C = 10^\circ$, $a = 200$ ft., find b, having given

$$\log 2 = {\cdot}3010300, \qquad \log 172{\cdot}64 = 2{\cdot}2371414,$$
$$L\sin 55^\circ = 9{\cdot}9133645, \qquad \log 172{\cdot}65 = 2{\cdot}2371666.$$

5. If in a triangle $b = 2{\cdot}25$ ft., $c = 1{\cdot}75$ ft., $A = 54^\circ$, find B and C, having given

$$\log 2 = {\cdot}301030, \qquad L\cot 27^\circ = 10{\cdot}292834,$$
$$L\tan 13^\circ\, 47' = 9{\cdot}389724, \qquad L\tan 13^\circ\, 48' = 9{\cdot}390270.$$

6. If the ratio of the lengths of two sides of a triangle is 9 : 7 and the included angle is 47° 25′, find the other angles, having given

$$\log 2 = {\cdot}3010300, \qquad L\tan 66^\circ\, 17'\, 30'' = 10{\cdot}3573942,$$
$$L\tan 15^\circ\, 53' = 9{\cdot}4541479, \qquad \text{diff. for } 1' = 4797.$$

7. An angle of a triangle is 60°, the area is $10\sqrt{3}$ and the perimeter is 20; find the remaining angles and the sides, having given

$$\log 2 = {\cdot}3010300, \qquad L\sin 49^\circ\, 6' = 9{\cdot}8784376,$$
$$\log 7 = {\cdot}8450980, \qquad L\sin 49^\circ\, 7' = 9{\cdot}8785470.$$

8. In a triangle ABC, it is given that $a = 10$ ft., $b = 9$ ft., $C = \tan^{-1}(\tfrac{4}{3})$; find c. If errors not greater than 1 in. each are made in measuring a and b, and an error not greater than 1° in measuring C, shew that the error in the calculated value of c will be less than 2·7 in.

9. In the ambiguous case, a, b, B being given, where $a > b$, if c, c' be the values of the third side, shew that $c^2 - 2cc'\cos 2B + c'^2 = 4b^2\cos^2 B$.

10. In the ambiguous case in which a, b, A are given, if one angle of one triangle be twice the corresponding angle of the other triangle, shew that

$$a\sqrt{3} = 2b\sin A, \quad \text{or} \quad 4b^3\sin^2 A = a^2(a + 3b).$$

11. The base of a triangle is equal to its altitude, and the two other sides are of known length; determine the remaining parts of the triangle by formulae adapted to logarithmic calculation. Shew that the ratio of the given sides must lie between $\frac{1}{2}(\sqrt{5} - 1)$ and $\frac{1}{2}(\sqrt{5} + 1)$.

12. A triangular piece of ground is 90 yards in its longest side, and 100 yards in the sum of the other two sides, and one of its angles is 46°. Determine the other angles, having given

$$L\tan 23^\circ = 9{\cdot}6278519,$$
$$L\tan 13^\circ\, 15' = 9{\cdot}3719333, \qquad L\tan 13^\circ\, 16' = 9{\cdot}3724992.$$

13. An angle of a triangle is 36°, the opposite side is 4, and the altitude $\sqrt{5} - 1$; solve the triangle.

14. Shew that it is impossible to construct a triangle out of the perpendiculars from the angles of a triangle on the sides if any side is $<\frac{1}{4}(3-\sqrt{5})\times$ perimeter; and it is certainly possible to construct such a triangle if each side is $>\frac{1}{6}$ perimeter.

15. If a triangle be solved from the parts $C=75°$, $b=2$, $c=\sqrt{6}$, shew that an error of $10''$ in the value of C would cause an error of about $3''\cdot 44$ in the calculated value of B.

16. Having given the mean side of a triangle whose sides are in A.P., and the angle opposite it, investigate formulae for solving the triangle, and find the greatest possible value of the given angle. Solve the triangle when the mean side is 542 feet, and the opposite angle is 59° 59′ 59″.

17. Solve a triangle, having given the length of the bisector of a side, and the angles into which this divides the vertical angle.

18. Solve a triangle, having given one side, the angle opposite it, and the perpendicular from that angle on the side.

19. A triangle is solved from the given parts a, b, A. If the values of a, b are affected by small errors x, y respectively, find the consequent error in the value of the perpendicular from A on the opposite side, and prove that this error is zero if $x\sin^2 B\cos C=y(\sin^2 B-\sin^2 C)$.

20. A lighthouse is seen N. 20°. E. from a vessel sailing S 25°. E and a mile further on it appears due N. Determine its distance at the last observation correctly to a yard, having given

$$L\sin 20°=9\cdot 534052 \qquad \log 2=\cdot 3010300,$$
$$\log 206=2\cdot 313867, \qquad \log 207=2\cdot 315900.$$

21. A cliff with a tower on its edge is observed from a boat at sea, the elevation of the top of the tower is 30°; after rowing towards the shore a distance of 500 yards in the plane of the first observation, the elevations of the top and bottom of the tower are 60° and 45° respectively; find the heights of the cliff and tower.

22. A is the foot of a vertical pole, B and C are due east of A, and D is due south of C. The elevation of the pole at B is double that at C, and the angle subtended by AB at D is $\tan^{-1}\frac{1}{8}$, also $BC=20$ ft., $CD=30$ ft.; find the height of the pole.

23. From a certain station the angular elevation of a mountain peak in the north-east is observed to be α. A hill in the east-south-east whose height above the station is known to be h, is then ascended, and the mountain peak is now seen in the north at an elevation β. Prove that the height of its summit above the first station is $h\sin\alpha\cos\beta\operatorname{cosec}(\alpha-\beta)$.

24. A train travelling on one of two straight intersecting railways subtends at a certain station on the other line an angle α, when the front of

the first carriage, and an angle α' when the end of the last, reaches the junction. Prove that the two lines are inclined to each other at an angle θ determined by $2\cot\theta = \cot\alpha \sim \cot\alpha'$.

25. A cylindrical tower stands on a horizontal plain; an eye in the plain views the visible arc of the rim of the upper end of the tower. If α, α', α'' be the angular elevations of either end of such arc above the plain, when the eye is at distances c, c', c'' respectively, prove that

$$(c'^2 - c''^2)\cot^2\alpha + (c''^2 - c^2)\cot^2\alpha' + (c^2 - c'^2)\cot^2\alpha'' = 0.$$

26. A balloon was observed in the N.E. at an elevation α; ten minutes afterwards it was found to be due N. at an elevation β. The rate at which the balloon was descending was afterwards ascertained to be six miles an hour; shew that its horizontal motion, supposed uniform, was at the rate of $\dfrac{6}{\sqrt{2}\tan\alpha - \tan\beta}$ miles an hour, the wind at the time being in the East.

27. I observe the angular elevation of the summits of two spires which appear in a straight line to be α, and the angular depressions of their reflexions in still water to be β and γ. If the height of my eye above the level of the water be c, then the horizontal distance between the spires is

$$\frac{2c\cos^2\alpha\sin(\beta-\gamma)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}.$$

28. The angular elevation of a tower at a place A due south of it is 30°, and at a place B, due west of A and at a distance a from it, the elevation is 18°; shew that the height of the tower is $\dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

29. A tower 51 feet high has a mark at a height of 25 feet from the ground; find at what distance the two parts subtend equal angles to an eye at the height of 5 feet from the ground.

30. A person on a level plain, on which stands a tower surmounted by a spire, observes that when he is a feet distant from the foot of the tower its top is in a line with that of a mountain. From a point b feet further from the tower he finds that the spire subtends at his eye the same angle as before, and has its top in a line with that of the mountain; shew that if the height of the tower above the horizontal plane through the observer's eye be c feet, the height of the mountain above that plane will be $\dfrac{abc}{c^2 - a^2}$ feet.

31 A man, 5 feet high, standing at the base of a pyramid whose base is square, sees the sun disappear over one of the edges, half-way along it. Shew that if a and b are the distances of the man from the two nearest corners, and θ is the altitude of the sun, the height of the pyramid is

$$10 + \tan\theta\sqrt{\tfrac{1}{2}(5a^2 - 2ab + b^2)} \text{ feet.}$$

32. From the top of a hill the depression of a point on the plain below is 30°, and from a spot three-quarters of the way down, the depression of the same point is 15°; find within 1′ the inclination of the hill.

33. *ABCD* is the rectangular floor of a room whose length *AB* is a feet. Find its height, which at C subtends at A an angle α, and at B an angle β. If $a=48$ ft., $\alpha=18°$, $\beta=30°$, prove that the height is 18 ft. 10 in. nearly.

34. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Shew that, if the distance of the observer from the foot of the hill be c, the height of the tower is $\dfrac{bc \sin \alpha}{a+b+c \cos \alpha}$.

35. A person standing between two towers observes that they subtend angles each equal to α, and on walking a feet along a straight path inclined at an angle γ to the line joining the towers, he finds that they subtend angles each equal to β; prove the following equations for determining the heights of the towers, $hh'(\cot^2\beta-\cot^2\alpha)=a^2$, $(h'-h)(\cot^2\beta-\cot^2\alpha)=2a\cot\alpha\cos\gamma$.

36. From a hill-top the angles of depression (α, β) of two piers of a bridge are observed, and the distance a between the piers subtends an angle θ at the point of observation; prove that the height of the hill is

$$\tfrac{1}{2}a \cot\phi \sec\tfrac{1}{2}\theta \sqrt{\sin\alpha\sin\beta},$$

where $$\cos\phi = 2\cos\tfrac{1}{2}\theta \,.\, \sqrt{\sin\alpha\sin\beta} \,.\, (\sin\alpha+\sin\beta)^{-1}.$$

37. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye, and that the angles of depression of their bases are α, α', α''; prove that, c, c', c'' being the heights of the towers,

$$\frac{\sin(\alpha'-\alpha'')}{c\sin\alpha}+\frac{\sin(\alpha''-\alpha)}{c'\sin\alpha'}+\frac{\sin(\alpha-\alpha')}{c''\sin\alpha''}=0.$$

38. A gun is fired from a fort, and the intervals between seeing the flash and hearing the report at two stations B, C are t, t' respectively; D is a point in the same straight line with BC, at a known distance a from A; prove that if $BD=b$, and $CD=c$, the velocity of sound is $\left\{\dfrac{(b-c)(a^2-bc)}{bt'^2-ct^2}\right\}^{\frac{1}{2}}$. Examine the case when $a^2=bc$.

39. From a point on a hill-side of constant inclination, the angle of elevation of the top of an obelisk on its summit is observed to be α, and a feet nearer to the top of the hill to be β; shew that, if h be the height of the obelisk, the inclination of the hill to the horizon will be

$$\cos^{-1}\left\{\frac{a}{h}\cdot\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right\}.$$

40. On the top of a spherical dome stands a cross; at a certain point the elevation of the cross is observed to be α, and that of the dome to be β; at a

distance a nearer the dome the cross is seen just above the dome, when its elevation is observed to be γ; prove that the height of the centre of the dome above the ground is $\frac{a \sin\gamma}{\sin(\gamma-\alpha)} \cdot \frac{\sin\alpha\cos\gamma - \cos\alpha\sin\beta}{\cos\gamma - \cos\beta}$.

41. At noon on a certain day the sun's altitude is α. A man observes a circular opening in a cloud which is vertically above a place at a distance a due south of him; he finds that the opening subtends an angle 2θ at his eye, and that the bright spot on the ground subtends an angle 2ϕ. Shew that if x is the height of the cloud

$$x^2(\cot^2\alpha\tan^2\phi - \tan^2\theta) - 2ax\cot\alpha\tan^2\phi + a^2(\tan^2\phi - \tan^2\theta) = 0.$$

42. From a point on the sloping face of a hill two straight paths are drawn, one in a vertical plane due South, the other in a vertical plane at right angles to the former, due East; these paths make with one another an angle α, and their lengths measured to the horizontal road at the foot of the hill are respectively a and b. Shew that the hill is inclined to the horizontal at an angle $\sin^{-1}\left(\frac{a^2+b^2-2ab\cos\alpha}{ab\sin\alpha\tan\alpha}\right)^{\frac{1}{2}}$.

43. The breadth of a straight river is calculated by measuring a base of length a along one side of the river and observing the angles made with this base by lines joining its extremities to a mark on the opposite bank. If the instrument by which the angles are measured gives each a value which is $(1+n)$ times the true value, n being very small, shew that the error in the computed breadth is nearly equal to $na \cdot \frac{\beta\sin^2\alpha - \alpha\sin^2\beta}{\sin^2(\alpha-\beta)}$; α, β being the circular measures of the above angles.

44. An observer from the deck of a ship, 20 feet above the sea, can just see the top of a distant lighthouse, and on ascending to the mast-head, where he is 80 feet above deck, he sees the door which he knows to be one-fourth of the height of the lighthouse above the level of the sea; find his distance from the lighthouse, and its height, assuming the earth to be a sphere of 4000 miles radius.

45. Three vertical posts are placed at intervals of one mile along a straight canal, each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point eight inches below the top; find to the nearest mile the radius of the earth.

46. Borings are made at three points A, B, C in a horizontal plane, and the depths at which gault is found are a, b, c respectively; also $AB=h$, $BC=k$, $ABC=\alpha$. If the upper surface of the gault be a plane, shew that its inclination ϕ to the horizon is given by

$$\tan^2\phi = \left\{\frac{(a-b)^2}{h^2} - 2\frac{(a-b)(c-b)}{hk}\cos\alpha + \frac{(c-b)^2}{k^2}\right\}\operatorname{cosec}^2\alpha.$$

47. The angular elevation of a column as viewed from a station due north of it being α, and as viewed from a station due east of the former station and at a distance c from it being β, prove that the height of the tower is

$$\frac{c \sin \alpha \sin \beta}{\{\sin (\alpha - \beta) \sin (\alpha + \beta)\}^{\frac{1}{2}}}.$$

48. A lighthouse stands 9 miles due N. of a port from which a yacht sails in a direction E.N.E., until the lighthouse is N.W. of her, when she tacks and sails towards the lighthouse until the port is S.W. of her, when she tacks again and sails into port. Shew that the length of the cruise is 16 miles nearly.

49. A circular pond of radius a is surrounded by a gravel walk of uniform width b, and the whole is enclosed by a fence of height d. A person of height h stands just inside the fence. Shew that the portion of the fence whose highest points can be seen by reflection from the water is $\frac{1}{n}$th, where

$$\frac{1}{n} = \frac{2}{\pi} \cos^{-1} \left\{ \frac{h+d}{2\sqrt{hd}} \frac{\sqrt{b^2+2ab}}{a+b} \right\},$$

provided $\qquad h < d(1+2a/b)$, and $> \dfrac{d}{1+2a/b}$.

50. The width of a croquet-hoop, the thickness of its wires, and the diameter of a ball are given; the ball being in a given position, shew how to find the conditions that it may just be possible for it to go through the hoop (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection.

51. Three mountain peaks, A, B, C, appear to an observer to be in a straight line, when he stands at each of two places P and Q, in the same horizontal line; the angle subtended by AB and BC at each place is α, and the angles AQP, CPQ are ϕ and ψ respectively.

Prove that the heights of the mountains are as

$$\cot 2\alpha + \cot \psi : \tfrac{1}{2} (\cot \alpha + \cot \psi)(\cot \alpha + \cot \phi) \tan \alpha : \cot 2\alpha + \cot \phi,$$

and that if QB cut AC in D, $AC = CD \sin 2\alpha (\cot \psi + \cot 2\alpha)$.

52. A man standing at a distance c from a straight line of railway sees a train standing upon the line, having its nearer end at a distance a from the point in the railway nearest him. He observes the angle α, which the train subtends, and thence calculates its length. If in observing the angle α he makes a small error θ, prove that the error in the calculated length of the train has to its true length a ratio $\dfrac{c\theta}{\sin \alpha (c \cos \alpha - a \sin \alpha)}$.

53. The height h of a mountain, whose summit is A, is to be determined from the observed values of a horizontal base line $BC\,(a)$, the angles ABC, ACB, and the angle (z) which AB makes with the vertical. Shew that

$$h = \frac{a \cos z \sin C}{\sin (B+C)}.$$

If h be known approximately, shew that the best direction of BC in order that an error in measuring C may have least effect on the accuracy of the above value of h, is given by $B=2\tan^{-1}\left(\dfrac{a\cos z-h}{a\cos z+h}\right)$.

54. Three vertical flag-staffs stand on a horizontal plane. At each of the points A, B and C in the horizontal plane, the tops of two of them are seen in the same straight line, and these straight lines make angles α, β, γ with the horizon. The plane containing the tops makes an angle θ with the horizon. Prove that their lengths are $BC/(\sqrt{\cot^2\beta-\cot^2\theta}+\sqrt{\cot^2\gamma-\cot^2\theta})$, and two similar expressions. Explain how the signs of the roots must be taken.

55. A tower AB stands on a horizontal plane and supports a spire BC. An observer at a place E on a mountain, whose side may be treated as an inclined plane, observes that AB, BC each subtend an angle α at his eye; he then moves to a place F, measuring the distance $EF\,(=2a)$, and observes that AB, BC again subtend angles α at his eye; he then measures the angles $AFE\,(=\beta)$ and $CFE\,(=\gamma)$. Shew that if x and y are the heights of AB, BC respectively,

$$x\cos\beta=y\cos\gamma=a\left\{1-\frac{\cos\beta\cos\gamma\cos^2\alpha}{\cos^2\frac{1}{2}(\beta+\gamma)\cos^2\frac{1}{2}(\beta-\gamma)}\right\}^{\frac{1}{2}}.$$

Also if G is the middle point of EF, and H is the point on the line of greatest slope through G, at which AB, BC subtend an angle δ, and GH is measured $(=b)$, prove that the inclination θ of the mountain to the horizon is given by

$$\left\{\frac{x^2y^2}{(x-y)^2}-\left(\frac{a^2+b^2}{2b}\right)^2\right\}^{\frac{1}{2}}\sin\theta+\frac{a^2+b^2}{2b}\cos\theta=\frac{xy(x+y)\sin2\delta}{x^2+y^2-2xy\cos2\delta}.$$

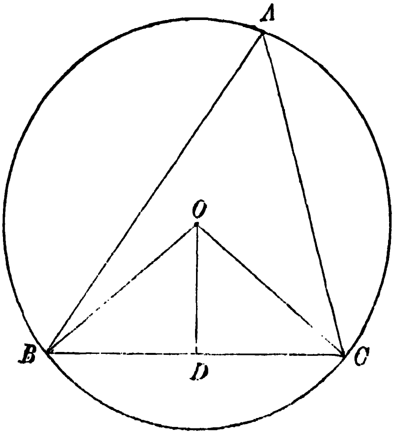
CHAPTER XII.

PROPERTIES OF TRIANGLES AND QUADRILATERALS.

150. In this Chapter we shall for the most part assume without proof the theorems in Euclidean Geometry which are necessary for our purpose, referring to works on pure Geometry for the investigation of those theorems.

The circumscribed circle of a triangle.

151. We have already, in Art. 120, obtained the formula $R = \frac{1}{2}a \operatorname{cosec} A$, for the radius of the circle circumscribing a triangle, or as it is now frequently called, the *circum-circle.* This formula may also be obtained as follows:



Let O be the circum-centre; draw OD perpendicular to the side BC of the triangle ABC, then D is the middle point of BC, and the angle $BOD = A$.

Since $BD = OB \sin BOD$ we have

$$\tfrac{1}{2}a = R \sin A, \text{ or } R = \tfrac{1}{2}a \operatorname{cosec} A \text{..............(1)}.$$

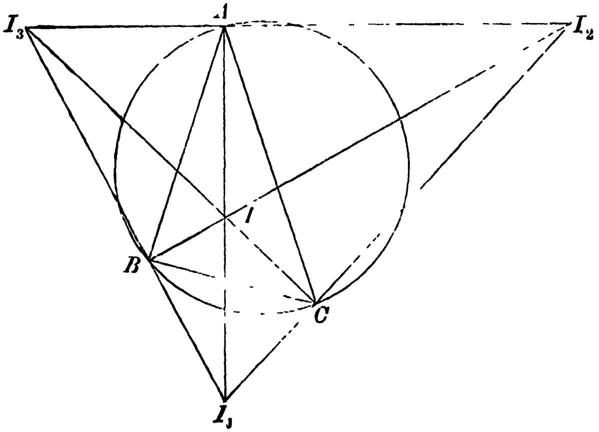
If S denote the area of the triangle ABC, we have

$S = \frac{1}{2}bc \sin A$, thus we have the expression $R = abc/4S$...(2).

Also $$OD = OB \cos A = R \cos A.$$

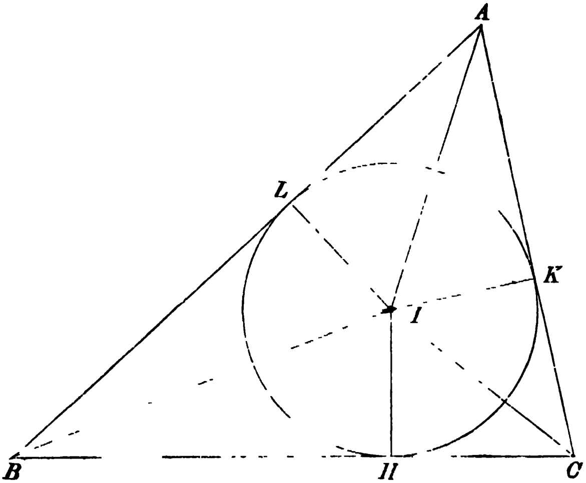
The inscribed and escribed circles of a triangle.

152. We know that four circles can be drawn touching the three sides of a triangle; the inscribed circle, or in-circle, touches each side internally, let I be its centre; the escribed circles each touch one side of the triangle and the other two sides produced, let I_1, I_2, I_3 be the centres of these circles; we know that IA, IB, IC bisect the angles A, B, C, respectively, and that IA bisects the angle A, and I_1B, I_1C bisect the angles B, C, externally; it follows therefore that AI_1, BI_2, CI_3 are the perpendiculars from I_1, I_2, I_3, on the opposite sides of the triangle $I_1I_2I_3$, and that I is the orthocentre of the triangle $I_1I_2I_3$.



The circum-circle of the triangle ABC is the nine-point circle of the triangle $I_1I_2I_3$,* and therefore passes through the middle points of the sides I_2I_3, I_3I_1, I_1I_2, and also through the middle points of II_1, II_2, II_3

153. Let H, K, L be the points of contact of the in-circle of the triangle ABC, with the sides BC, CA, AB, respectively.



Then $$\Delta IBC + \Delta ICA + \Delta IAB = S.$$

Now $\Delta IBC = \frac{1}{2} IH . BC = \frac{1}{2} ra$, $\Delta ICA = \frac{1}{2} rb$, $\Delta IAB = \frac{1}{2} rc$, where r denotes the radius of the in-circle, hence

$\frac{1}{2} r (a+b+c) = S$, whence we have the formula $r = S/s \ldots (3)$, for the radius of the in-circle.

Also $$a = BH + HC = r(\cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C),$$

hence $$r = a \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \sec \tfrac{1}{2} A \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

another expression for r, which might of course be deduced from (3).

Combining the formulae (1) and (4) we have the symmetrical expression $$r = 4R \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Again, since $AK + BC = \frac{1}{2}(BC + CA + AB)$,

we have $$AK = AL = s - a,$$

and similarly $BH = BL = s - b$, $CH = CK = s - c$,

hence since $r = AK \tan \frac{1}{2} A = BH \tan \frac{1}{2} B = CK \tan \frac{1}{2} C$,

we obtain the expressions

$$r = (s - a) \tan \tfrac{1}{2} A = (s - b) \tan \tfrac{1}{2} B = (s - c) \tan \tfrac{1}{2} C \ldots\ldots (6),$$

which may also be deduced from (3) or (4).

154. Expressions corresponding to those of the last Article may be found for the radii r_1, r_2, r_3 of the escribed circles.

Let H_1, K_1, L_1 be the points of contact of the circle whose centre is I_1, with the sides of the triangle ABC. Then

$$\Delta I_1AB + \Delta I_1AC - \Delta I_1BC = S, \text{ therefore } \tfrac{1}{2}r_1(b+c-a) = S,$$

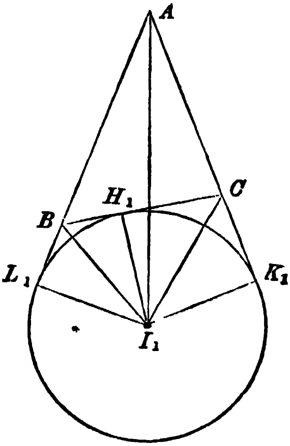
thus we have the formulae

$$r_1 = \frac{S}{s-a}, \quad r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c} \quad \text{............(7)},$$

for the radii of the escribed circles.

Also $\quad a = BH_1 + H_1C = r_1(\tan\tfrac{1}{2}B + \tan\tfrac{1}{2}C)$,

therefore $\quad r_1 = a\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sec\tfrac{1}{2}A$ ....(8),



whence we obtain the formula

$$r_1 = 4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C \quad \text{..............(9)},$$

with corresponding expressions for r_2 and r_3.

Again, since

$$BH_1 = BL_1, \text{ and } CH_1 = CK_1, \text{ and } AK_1 = AL_1,$$

we find $\quad BH_1 = s-c, \quad CH_1 = s-b, \quad AK_1 = AL_1 = s,$

thus we obtain the formulae

$$r_1 = s\tan\tfrac{1}{2}A = (s-c)\cot\tfrac{1}{2}B = (s-b)\cot\tfrac{1}{2}C \text{......(10)}.$$

EXAMPLES.

(1) *Prove that*

$$r_1+r_2+r_3-r=4R,$$
$$r_2r_3+r_3r_1+r_1r_2=S^2/r^2,$$
$$r_1^{-1}+r_2^{-1}+r_3^{-1}=r^{-1}.$$

(2) *Prove the following formulae for the sides and angles of a triangle, in terms of the radii of the escribed circles:*

$$(\alpha)\quad a=\frac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}},\qquad (\beta)\quad \sin\tfrac{1}{2}A=\frac{r_1}{\sqrt{(r_1+r_2)(r_1+r_3)}},$$
$$(\gamma)\quad \sin A=2r_1\frac{\sqrt{r_2r_3+r_3r_1+r_1r_2}}{(r_1+r_2)(r_1+r_3)}.$$

(3) *Prove that* $$R=\frac{1}{4}\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}.$$

(4) *Prove that* $$16R^2rr_1r_2r_3=a^2b^2c^2.$$

(5) *Prove that* $$\cos A=\frac{2R+r-r_1}{2R}.$$

(6) *If the escribed circle which touches* a *is equal to the circum-circle, prove that* $\cos A=\cos B+\cos C$.

(7) *Prove that* $r_1(r_2+r_3)\operatorname{cosec} A=r_2(r_3+r_1)\operatorname{cosec} B=r_3(r_1+r_2)\operatorname{cosec} C$.

(8) *If* a, a_1, a_2, a_3 *are the distances of the centres of the inscribed and escribed circles from* A, *and* p *is the perpendicular from* A *on* BC, *prove that*

$$(a)\quad aa_1a_2a_3=4R^2p^2,$$
$$(b)\quad a^2+a_1^2+a_2^2+a_3^2=16R^2,$$
$$(c)\quad a^{-2}+a_1^{-2}+a_2^{-2}+a_3^{-2}=4p^{-2}.$$

(9) *Shew that the area of the triangle formed by joining the centres of the escribed circles is* $\frac{abc}{2r}$, *or* $8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

(10) *Shew that the radius of the circle round any of the four triangles formed by joining the centres of the inscribed and escribed circles is double of* R.

(11) *Prove that the areas* $I_1I_2I_3$, I_2I_3I, I_3I_1I, I_1I_2I *are inversely as* r, r_1, r_2, r_3.

(12) *Prove that*

$$(a)\quad \frac{I_2I_3^2}{r_2r_3}+\frac{I_3I_1^2}{r_3r_1}+\frac{I_1I_2^2}{r_1r_2}=8\frac{R}{r},$$
$$(b)\quad r^3 . II_1 . II_2 . II_3=IA^2 . IB^2 . IC^2.$$

(13) *If* d_1, d_2, d_3 *be the distances of* I *from the angular points of a triangle, shew that* $\frac{d_1d_2d_3}{abc}=\frac{r}{s}$.

(14) *If* a′, b′, c′ *are the sides of the triangle formed by joining the points of contact* H_1, H_2, H_3 *of the escribed circles, shew that* $\frac{a^2-a'^2}{a}=\frac{b^2-b'^2}{b}=\frac{c^2-c'^2}{c}$.

(15) *Prove that the sides of the triangle formed by joining the centres of the circles* BOC, COA, AOB *are as* $\sin 2A : \sin 2B : \sin 2C$.

(16) *Prove that the circum-circles of the two triangles in the ambiguous case, when* a, b, B *are given, are equal in magnitude; shew also that the distance between their centres is* $(b^2 \operatorname{cosec}^2 B - a^2)^{\frac{1}{2}}$.

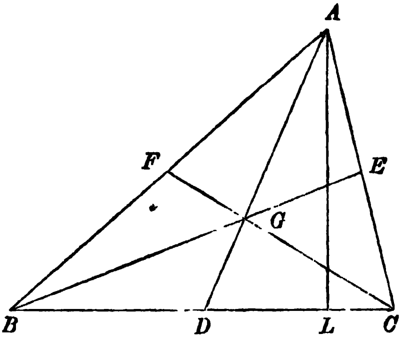
(17) *In the ambiguous case of the solution of a triangle, prove that the distance of the points of contact of the inscribed circles with the greater of the two given sides is equal to half the difference of the values of the third side.*

(18) *If* ρ_1, ρ_2, ρ_3 *be the radii of the circles described about* IBC, ICA, IAB, *prove that* $4R^3 - R(\rho_1^2 + \rho_2^2 + \rho_3^2) - \rho_1\rho_2\rho_3 = 0$.

(19) *Prove that the radii of the escribed circles of a triangle are the roots of the cubic* $x^3 - x^2(4R + r) + xs^2 - rs^2 = 0$.

The medians.

155. The lines AD, BE, CF, joining the angular points of a triangle to the middle points of the opposite sides, are called the *medians*. The length of AD is given by the well-known geometrical theorem $AB^2 + AC^2 = 2(AD^2 + BD^2)$, thus the squares of their lengths are given by



$$m_1^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2, \quad m_2^2 = \tfrac{1}{2}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}b^2,$$
$$m_3^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2 \quad \text{........(11)}.$$

Let M_1 denote the angle ADC, then

$$\cot M_1 = DL/AL = \tfrac{1}{2}(BL - CL)/AL,$$

where AL is perpendicular to BC, therefore M_1 is given by

$$\cot M_1 = \tfrac{1}{2}(\cot B - \cot C) \quad \text{....... ....(12)}.$$

The point G, where the medians intersect one another, is called the *centroid* of the triangle. It is well known that G divides each of the medians in the ratio 2 : 1.

Examples.

(1) *Prove that* $\cot \mathrm{AGF} + \cot \mathrm{BGD} + \cot \mathrm{CGE} = \cot \mathrm{A} + \cot \mathrm{B} + \cot \mathrm{C}$.

(2) *If* α, β, γ *are the centres of the circles* BGC, CGA, AGB, *and* Δ, Δ' *are the areas of the triangles* ABC, $\alpha\beta\gamma$, *prove that* $48\Delta\Delta' = (\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2)^2$.

(3) *If* $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ *be the radii of the circles* BGC, CGA, AGB, *prove that*

$$\frac{\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}{\mathrm{R}_1^2} + \frac{\mathrm{b}^2(\mathrm{c}^2-\mathrm{a}^2)}{\mathrm{R}_2^2} + \frac{\mathrm{c}^2(\mathrm{a}^2-\mathrm{b}^2)}{\mathrm{R}_3^2} = 0.$$

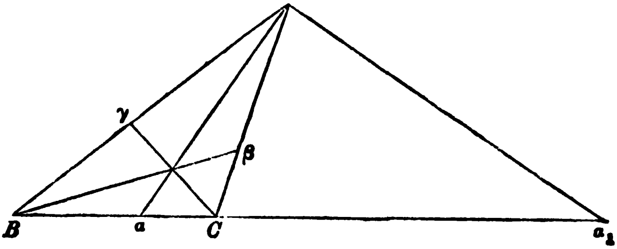
(4) *If the angles* BAD, CBE, ACF *are* α, β, γ, *and the angles* CAD, ABE, BCF *are* α', β', γ', *prove that*

$$\cot \alpha + \cot \beta + \cot \gamma = \cot \alpha' + \cot \beta' + \cot \gamma'.$$

The bisectors of the angles.

156. Let α and α_1 be the points in which the internal and external bisectors of the angle A meet the opposite side BC. Let f, g, h be the lengths of the internal bisectors $A\alpha$, $B\beta$, $C\gamma$, and f', g', h' the lengths of the external bisectors $A\alpha_1$, $B\beta_1$, $C\gamma_1$. To find the positions of α and α_1, we have $B\alpha/C\alpha = BA/CA = B\alpha_1/C\alpha_1$, whence

$$B\alpha = \frac{ac}{b+c}, \quad C\alpha = \frac{ab}{b+c}, \quad B\alpha_1 = \frac{ac}{c-b}, \quad C\alpha_1 = \frac{ab}{c-b}.$$



To find the lengths f, f', we have

$$\Delta AB\alpha + \Delta AC\alpha = S = \Delta A\alpha_1 B - \Delta A\alpha_1 C,$$

hence $$f(b+c)\sin\tfrac{1}{2}A = f'(c-b)\cos\tfrac{1}{2}A = 2S,$$

therefore f and f' are given by

$$f = \frac{2bc}{b+c}\cos\tfrac{1}{2}A, \quad f' = \frac{2bc}{c-b}\sin\tfrac{1}{2}A \text{ ....(13).}$$

EXAMPLES.

(1) *If* α, β, γ *are the angles that* $\mathrm{A}\alpha$, $\mathrm{B}\beta$, $\mathrm{C}\gamma$ *make with the sides* a, b, c, *shew that* $\mathrm{a}\sin 2\alpha + \mathrm{b}\sin 2\beta + \mathrm{c}\sin 2\gamma = 0$

(2) *If* f_1, g_1, h_1 *are the lengths of the bisectors of the angles, produced to meet the circum-circle, shew that*

$$\mathrm{f}^{-1}\cos\tfrac{1}{2}\mathrm{A} + \mathrm{g}^{-1}\cos\tfrac{1}{2}\mathrm{B} + \mathrm{h}^{-1}\cos\tfrac{1}{2}\mathrm{C} = \mathrm{a}^{-1} + \mathrm{b}^{-1} + \mathrm{c}^{-1},$$

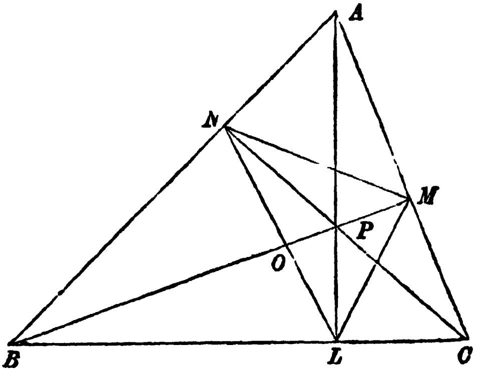
and

$$\mathrm{f}_1\cos\tfrac{1}{2}\mathrm{A} + \mathrm{g}_1\cos\tfrac{1}{2}\mathrm{B} + \mathrm{h}_1\cos\tfrac{1}{2}\mathrm{C} = \mathrm{a} + \mathrm{b} + \mathrm{c}.$$

(3) *Prove that* $\alpha\beta$ *cuts* $\mathrm{C}\gamma$ *in the ratio* $2\mathrm{c} : \mathrm{a}+\mathrm{b}$.

The pedal triangle.

157. The triangle LMN formed by joining the feet of the perpendiculars AL, BM, CN, from A, B, C, on the opposite sides, is called the *pedal triangle* of A, B, C. Let P be the orthocentre of the triangle ABC, then since PMA, PNA are right angles, a circle whose diameter is PA circumscribes $PMAN$, hence MN is equal to PA multiplied by the sine of the angle in the segment MN, or $MN = PA\sin A$; now if O is the centre of the circum-circle, and OD is perpendicular to BC, it is well known that $AP = 2OD$, and we have shewn in Art. 151 that this is equal to $2R\cos A$: hence $MN = 2R\sin A\cos A = a\cos A$. Also



the angles PLM, PLN are each the complement of A, or $MLN=\pi-2A$; the sides and angles of the pedal triangle are therefore respectively

$$\left.\begin{array}{lll} a\cos A, & b\cos B, & c\cos C \\ \pi-2A, & \pi-2B, & \pi-2C \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots(14).$$

It should be remarked that ABC is the pedal triangle of $I_1I_2I_3$. The pedal triangle of LMN is called the second pedal triangle of ABC, and so on.

We have assumed that the triangle is acute-angled; if the angle A is obtuse, it can be easily shewn that the angles of the pedal triangle are $2A-\pi$, $2B$, $2C$, and that the sides are $-a\cos A$, $b\cos B$, $c\cos C$.

EXAMPLES.

(1) *Prove that the radius of the circle inscribed in the triangle* LMN *is* $2R\cos A\cos B\cos C$.

(2) *If* α, β, γ *are the diameters of the circles* MPN, NPL, LPM, *shew that*

$$\frac{\beta\gamma}{bc}+\frac{\gamma\alpha}{ca}+\frac{\alpha\beta}{ab}=1.$$

(3) *Prove that if* r', r_1', r_2', r_3' *are the radii of the inscribed and escribed circles of the pedal triangle, then* $\dfrac{r_1'r_2'r_3'}{r'}=\dfrac{rr_1r_2r_3}{R^2}$.

(4) *If* AL, BM, CN *meet the circum-circle in* L′, M′, N′, *shew that*

$$\frac{AL'}{AL}+\frac{BM'}{BM}+\frac{CN'}{CN}=4.$$

The distances between special points.

158. Let P be the orthocentre, O the centre of the circum-circle, I of the in-circle, I_1 of one of the escribed circles, G the centroid, and U the centre of the nine-point circle of the triangle ABC. According to Euler's well-known theorem, the three points O, G, P lie on a straight line, and $PG=2OG$; the point U is also on OP, at its middle point. Each of the angles IAO, IAP is equal to $\frac{1}{2}(B\sim C)$; also $AO=R$, $AP=2R\cos A$,

$$AI=r\operatorname{cosec}\tfrac{1}{2}A=4R\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C,\quad AI_1=4R\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

We can now find expressions for the distances of the points O, I, P, I_1, U from one another.

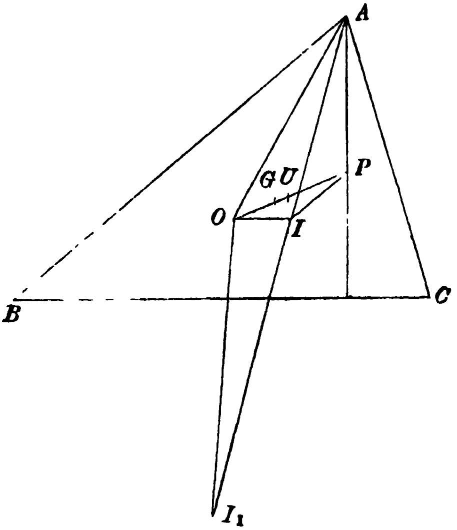
(1) *To find* OI $= \delta$. We have

$$\delta^2 = AO^2 + AI^2 - 2AO \,.\, AI \cos OAI,$$

hence

$$\delta^2 = R^2 (1 + 16 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C - 8 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}\overline{B-C})$$

or $$\delta^2 = R^2 (1 - 8 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C),$$



we thus obtain Euler's formula

$$\delta^2 = R^2 - 2Rr \text{ ....(15).}$$

(2) *To find* OI$_1 = \delta_1$. We have

$$\delta_1^2 = R^2 (1 + 16 \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C - 8 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \cos \tfrac{1}{2}\overline{B-C})$$

or $$\delta_1^2 = R^2 (1 + 8 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C),$$

which gives $$\delta_1^2 = R^2 + 2Rr_1 \text{ ....(16).}$$

(3) *To find* OP.

From the triangle OAP we have

$$OP^2 = OA^2 + AP^2 - 2OA \,.\, AP \cos OAP$$

or $$OP^2 = R^2 (1 + 4 \cos^2 A - 4 \cos A \cos \overline{B-C}),$$

which gives $$OP^2 = R^2 (1 - 8 \cos A \cos B \cos C) \text{ ....(17).}$$

(4) *To find* IP. We have

$$IP^2 = 4R^2\cos^2 A + 16R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C$$
$$- 16R^2\cos A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(B-C),$$

hence $IP^2 = 4R^2\{\cos^2 A + (1-\cos B)(1-\cos C) - \cos A\sin B\sin C$
$- \cos A(1-\cos B)(1-\cos C)\}$,

or $IP^2 = 4R^2\{(1-\cos A)(1-\cos B)(1-\cos C)$
$-\cos A\cos B\cos C\}$......(18),

or $IP^2 = 2r^2 - 4R^2\cos A\cos B\cos C$.

(5) *To find* IU. We have

$$IU^2 = \tfrac{1}{2}IP^2 + \tfrac{1}{2}IO^2 - \tfrac{1}{4}OP^2;$$

hence $IU^2 = r^2 + \frac{1}{2}R^2 - Rr - \frac{1}{4}R^2 = (\frac{1}{2}R - r)^2$;

hence $IU = \frac{1}{2}R - r$; in a similar manner it can be shewn that $I_1U = \frac{1}{2}R + r_1$; now $\frac{1}{2}R$ is the radius of the nine-point circle, hence the expressions we have obtained for IU, I_1U shew that *the inscribed and escribed circles touch the nine-point circle.* We have then a trigonometrical proof of Feuerbach's theorem, of which a considerable number of geometrical proofs have been given.

EXAMPLES.

(1) *If* t_1, t_2, t_3 *are the lengths of the tangents from the centres of the escribed circles to the circum-circle, prove that*

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{a+b+c}{abc}.$$

(2) *Prove that the area of the triangle* IOP *is*

$$-2R^2\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B).$$

(3) *Prove that* $GI^2 = \frac{16}{3}R^2\{\Sigma\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C - \frac{1}{12}\Sigma\sin^2 A\}$

and $GI^2 + 4Rr = \frac{1}{3}(bc+ca+ab) - \frac{1}{9}(a^2+b^2+c^2)$.

(4) *Prove that* $OP^2 = \dfrac{\Sigma a^2(a^2-b^2)(a^2-c^2)}{(4S)^2}$.

(5) *If* α, β, γ *be the distances of the centre of the nine-point circle from the angular points, and* g *its distance from the orthocentre, shew that*

$$\alpha^2 + \beta^2 + \gamma^2 + g^2 = 3R^2.$$

(6) *Prove that the nine-point circle does not cut the circum-circle unless the triangle is obtuse, and in that case they cut at an angle*

$$\cos^{-1}(1 + 2\cos A\cos B\cos C).$$

(7) *Shew that, if the distance between the orthocentre and the centre of the circum-circle is* $\frac{1}{2}a$, *the triangle is right-angled, or else* $\tan B \tan C = 9$.

(8) *If* Q *is the centre of the nine-point circle, shew that*

$$(QI_2 - QI_3)(QI_1 - QI) = b^2 - c^2.$$

(9) *If* OIP *is an equilateral triangle, shew that* $\cos A + \cos B + \cos C = \frac{3}{2}$.

(10) *If the centre of the in-circle be equidistant from the centre of the circum-circle and the orthocentre, prove that one angle of the triangle is* 60°.

Expressions for the area of a triangle.

159. A very large number of expressions for the area of a triangle, in terms of various lines and angles connected with the triangle, have been given. Large collections of such formulae will be found in *Mathesis*, Vol. III. and in the *Annals of Mathematics*, Vol. I. No. 6.

We give here a few of these expressions, leaving the verification of them as an exercise for the student.

(1) $\sqrt{rr_1r_2r_3}$, (2) $\sqrt{\frac{1}{2}Rp_1p_2p_3}$, (3) $\frac{4}{3}\sqrt{\sigma(\sigma-m_1)(\sigma-m_2)(\sigma-m_3)}$

where $2\sigma = m_1 + m_2 + m_3$.

(4) $\dfrac{s^2}{\Sigma\cot\frac{1}{2}A}$, (5) $\dfrac{f\cos\frac{1}{2}(B-C)+g\cos\frac{1}{2}(C-A)+h\cos\frac{1}{2}(A-B)}{2(f^{-1}\cos\frac{1}{2}A+g^{-1}\cos\frac{1}{2}B+h^{-1}\cos\frac{1}{2}C)}$,

(6) $r^2\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$, (7) $r^2\cot\frac{1}{2}A + 2Rr\sin A$,

(8) $r_2r_3\tan\frac{1}{2}A$, (9) $rr_1\dfrac{r_2-r_3}{b-c}$, (10) $r_1r_2\sqrt{\dfrac{4R-(r_1+r_2)}{r_1+r_2}}$.

Various properties of triangles.

160. If Q be any point in the plane of the triangle ABC, we have the identical relation $\triangle QBC + \triangle QCA + \triangle QAB = \triangle ABC$, the areas of the triangles with vertex Q being taken with the proper signs; for example, $\triangle QBC$ is negative when Q and A are on opposite sides of BC. By taking Q in various positions, we obtain various well-known relations between the angles of a triangle.

(1) Let Q be at O, the above relation becomes

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C,$$

since the angles BOC, COA, AOB are $2A$, $2B$, $2C$ respectively.

(2) Let Q be at I, we obtain the relation

$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C)+\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(C+A)+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A+B)$$
$$=2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

(3) Let Q be at U, we get

$$\sin A\cos(B-C)+\sin B\cos(C-A)+\sin C\cos(A-B)$$
$$=4\sin A\sin B\sin C.$$

161. The identical relation which holds between the six distances of any four points A, B, C, Q, in a plane, may be expressed in various forms.

(1) Using the equation $\triangle QBC+\triangle QCA+\triangle QAB=\triangle ABC$, and expressing each of the four triangles in terms of its sides, we have the required relation in a form involving four radicals.

(2) To obtain the same relation in a rationalised form, denote the angles BQC, CQA, AQB by α, β, γ respectively; then since $\alpha+\beta+\gamma=2\pi$, we have

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=0.$$

Now substituting for $\cos\alpha$ its value $(QB^2+QC^2-BC^2)/2QB.QC$ with the corresponding expressions for $\cos\beta$, $\cos\gamma$, we have the required relation.

162. Taking any general relation between the sides and angles of a triangle, another relation may be deduced, by replacing the sides and angles by the corresponding sides and angles of the pedal triangle. The sides and angles of this triangle are given in (14), and we may therefore replace a, b, c, in the given relation, by $a\cos A$, $b\cos B$, $c\cos C$, and the angles A, B, C by $\pi-2A$, $\pi-2B$, $\pi-2C$.

As an example of this transformation, we obtain from the known relation $a^2=b^2+c^2-2bc\cos A$, the new relation

$$a^2\cos^2A=b^2\cos^2B+c^2\cos^2C+2bc\cos B\cos C\cos 2A.$$

This method of transformation may be extended, by taking the nth pedal triangle, of which the sides are

$$(-1)^{n-1}a\cos A\cos 2A\cos 4A\ldots\cos 2^{n-1}A,$$
$$(-1)^{n-1}b\cos B\cos 2B\cos 4B\ldots\cos 2^{n-1}B,$$
$$(-1)^{n-1}c\cos C\cos 2C\ldots\cos 2^{n-1}C,$$

and the angles are

$$\tfrac{1}{3}(2^n+1)\pi-2^nA,\quad \tfrac{1}{3}(2^n+1)\pi-2^nB,\quad \tfrac{1}{3}(2^n+1)\pi-2^nC,$$

when n is odd, and

$$-\tfrac{1}{3}(2^n-1)\pi+2^nA,\quad -\tfrac{1}{3}(2^n-1)\pi+2^nB,\quad -\tfrac{1}{3}(2^n-1)\pi+2^nC,$$

when n is even.

Thus, in any relation between the sides and angles of a triangle, we are entitled to write $(-1)^{n-1}a\cos A\cos 2A\ldots\cos 2^{n-1}A$ for a, and $\tfrac{1}{3}(2^n+1)\pi-2^nA$ or $2^nA-\tfrac{1}{3}(2^n-1)\pi$ for A, according as n is odd or even, with corresponding expressions for the other sides and angles.

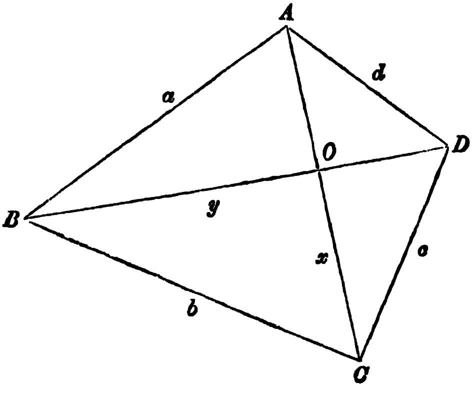
163. In any general relation between the sines and cosines of the angles of a triangle, we may substitute $pA+qB+rC$, $qA+rB+pC$, $rA+pB+qC$ for A, B, C respectively, where p, q, r are any numbers such that $p+q+r$ is of one of the forms $6n-1$, $6n+2$, where n is a positive integer, provided that when $p+q+r$ is of the form $6n-1$, the signs of all the sines are changed, and when $p+q+r$ is of the form $6n+2$, the signs of all the cosines are changed.

This theorem follows from the facts that in the first case the sum of the angles $2n\pi-(pA+qB+rC)$, $2n\pi-(qA+rB+pC)$, $2n\pi-(rA+pB+qC)$ is π, and in the latter case the sum of the three angles

$$(2n+1)\pi-(pA+qB+rC),\quad (2n+1)\pi-(qA+rB+pC),$$
$$(2n+1)\pi-(rA+pB+qC),\ \text{is}\ \pi.$$

Properties of quadrilaterals.

164. Let $ABCD$ be a convex quadrilateral; denote the sides AB, BC, CD, DA by a, b, c, d respectively, and the diagonals AC,



BD by x, y respectively; also let $A + C = 2\alpha$, and let ϕ be the angle between the diagonals.

We shall find an expression for the area S of the quadrilateral in terms of a, b, c, d, and α. We have

$$y^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C,$$

therefore $$ad\cos A - bc\cos C = \tfrac{1}{2}(a^2 + d^2 - b^2 - c^2),$$

also $$ad\sin A + bc\sin C = 2S;$$

square and add the corresponding sides of these equations, we get

$$a^2d^2 + b^2c^2 - 2abcd\cos 2\alpha = 4S^2 + \tfrac{1}{4}(a^2 + d^2 - b^2 - c^2)^2,$$

hence $16S^2 = 4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd\cos^2\alpha$,

or $16S^2 = \{(a + d)^2 - (b - c)^2\}\{(b + c)^2 - (a - d)^2\} - 16abcd\cos^2\alpha$;

hence $$S^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha \quad\text{......(19)},$$

where $$2s = a + b + c + d.$$

In the case of a quadrilateral inscribable in a circle we have $2\alpha = \pi$, thus

$$S^2 = (s - a)(s - b)(s - c)(s - d) \quad\text{............(20)}.$$

The expression (19) shews that the quadrilateral of which the sides are given has its area greatest when $\alpha = \frac{1}{2}\pi$, that is, when the quadrilateral can be inscribed in a circle.

The theorem (20) was discovered by *Brahmegupta*, a Hindoo Mathematician of the sixth century.

165. Expressions for the area of a quadrilateral can be found, which involve the lengths of the diagonals and the angle between them.

The area of the quadrilateral is the sum of the areas of the four triangles into which the diagonals divide it; the area of each of these triangles is half the product of the two segments of the diagonals which are sides of it, multiplied by $\sin\phi$; hence by addition we have

$$S = \tfrac{1}{2}xy\sin\phi \quad\text{........................(21)}.$$

Also

$2OA \,.\, OB\cos\phi = OA^2 + OB^2 - a^2$, $2OC \,.\, OD\cos\phi = OC^2 + OD^2 - c^2$,

$2OA \,.\, OD\cos\phi = d^2 - OA^2 - OD^2$, $2OB \,.\, OC\cos\phi = b^2 - OB^2 - OC^2$;

hence $$2xy\cos\phi = b^2 + d^2 - a^2 - c^2 \quad\text{................(22)},$$

therefore $$S = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\phi \quad\text{..............(23)},$$

and eliminating ϕ, we obtain Bretschneider's formula

$$S = \tfrac{1}{4}\{4x^2y^2 - (b^2 + d^2 - a^2 - c^2)^2\}^{\frac{1}{2}} \;\ldots\ldots\ldots\ldots(24),$$

which expresses the area in terms of the diagonals and the sides.

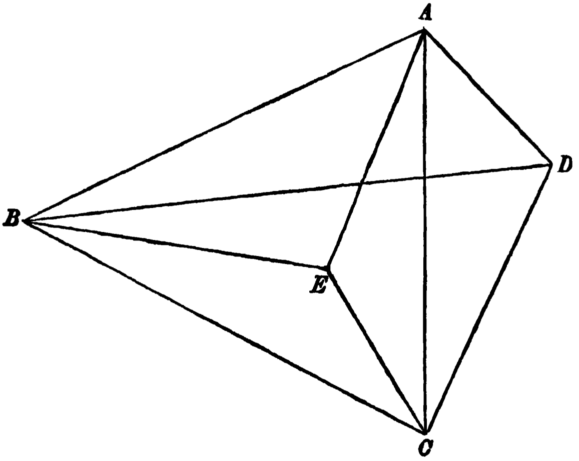
If a circle can be inscribed in the quadrilateral, we have $a+c=b+d$, hence the formulae (23), (24) become $S=\tfrac{1}{2}(ac-bd)\tan\phi$, and

$$S=\tfrac{1}{2}\{x^2y^2-(ac-bd)^2\}^{\frac{1}{2}}.$$

166. An expression may be found for the product of the diagonals of a quadrilateral, in terms of the sides and the cosine of the sum of two opposite angles.

Through B and C draw straight lines meeting in E, so that the angles CBE, BCE may be equal to the angles ABD, ADB, respectively. The triangles ECB, ABD are similar, hence

$$\frac{AD}{CE}=\frac{BD}{CB}=\frac{AB}{BE},$$



thus $AD.CB=BD.CE$. Also since the angles CBD, ABE are equal, and $AB:BE::BD:BC$, the triangles ABE and CBD are similar, therefore $AB.CD=BD.AE$.

Since $AC^2=AE^2+EC^2-2AE.EC\cos(A+C)$,

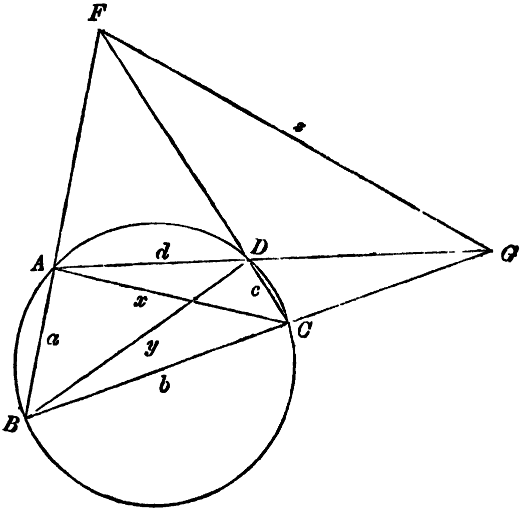
multiplying by BD^2, we have

$$x^2y^2=a^2c^2+b^2d^2-2abcd\cos 2\alpha \;\ldots\ldots\ldots\ldots(25).$$

If $2\alpha = \pi$, we have Ptolemy's theorem $xy = ac + bd$, for a quadrilateral inscribed in a circle.

If $2\alpha = \frac{1}{2}\pi$, we have $x^2y^2 = a^2c^2 + b^2d^2$, for a quadrilateral in which the sum of two opposite angles is a right angle.

167. In the case of a quadrilateral inscribed in a circle, the lengths of the diagonals x, y, and of the third diagonal, formed by joining the point of intersections of the sides a and c to that of b and d, may be found in terms of the sides.



Let FG be the third diagonal, and denote the lengths of AC, BD, FG by x, y, z respectively. We have

$$x^2 = a^2 + b^2 - 2ab\cos B$$

and

$$x^2 = c^2 + d^2 - 2cd\cos D,$$

hence

$$x^2\left(\frac{1}{ab} + \frac{1}{cd}\right) = \frac{a^2 + b^2}{ab} + \frac{c^2 + d^2}{cd},$$

hence

$$x^2 = (ac + bd)(ad + bc)/(ab + cd) \quad \ldots\ldots\ldots\ldots(26),$$

and similarly it may be shewn that

$$y^2 = (ac + bd)(ab + cd)/(ad + bc).$$

We have also

$$FA = AD \cdot \frac{\sin D}{\sin(A + D)} = \frac{dx}{y\cos D + x\cos A},$$

and similarly
$$FB = \frac{by}{y\cos D + x\cos A},$$

hence
$$\frac{FA}{dx} = \frac{FB}{by} = \frac{FB - FA}{by - dx} = \frac{a}{by - dx}$$

hence
$$FA \,.\, FB = \frac{a^2 bdxy}{(by - dx)^2};$$

it may be shewn in a similar manner that

$$GC \,.\, GB = \frac{b^2 acxy}{(ay - cx)^2}.$$

Now the square on FG is equal to the sum of the squares of the tangents from F and G to the circle (see McDowell's *Geometry*, p. 92), hence we have

$$z^2 = xy\left\{\frac{a^2 bd}{(by - dx)^2} + \frac{b^2 ac}{(ay - cx)^2}\right\}.$$

Now from the values found above, for x^2 and y^2, we have

$$\frac{x}{ad + bc} = \frac{y}{ab + cd} = \frac{by - dx}{a(b^2 - d^2)} = \frac{ay - cx}{b(a^2 - c^2)},$$

therefore substituting in the expression for z^2, we obtain

$$z^2 = (ad + bc)(ab + cd)\left\{\frac{bd}{(b^2 - d^2)^2} + \frac{ac}{(a^2 - c^2)^2}\right\} \ldots\ldots (27).$$

EXAMPLES.

(1) *If the quadrilateral is inscribed in a circle, shew that the radius of the circle is*

$$\frac{1}{4}\left\{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}\right\}^{\frac{1}{2}}.$$

(2) *Shew that the distance between the centre of a circle, of radius* r, *and the intersection of the diagonals of an inscribed quadrilateral is*

$$\frac{r}{(ab+cd)(ad+bc)}\left[(ac+bd)\left\{ac(b^2-d^2)^2 + bd(a^2-c^2)^2\right\}\right]^{\frac{1}{2}}.$$

(3) *Shew that the diagonals of a quadrilateral inscribed in a circle meet at an angle* $\cos^{-1}\frac{(a^2+c^2) \sim (b^2+d^2)}{2(ac+bd)}$ *or* $2\tan^{-1}\left\{\frac{(s-b)(s-d)}{(s-a)(s-c)}\right\}^{\frac{1}{2}}$, *and that the product of the segments of a diagonal is* $\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}$.

(4) *If* S *is the area of a quadrilateral inscribed in a circle, shew that the straight lines joining the middle points of the opposite sides meet at an angle*

$$\tan^{-1}\left\{\frac{4S}{(b^2 \sim d^2)(a^2 \sim c^2)} \cdot \frac{(ad+bc)(ab+cd)}{ac+bd}\right\}.$$

(5) *If* E, F, G *are the intersections of pairs of the diagonals of a quadrilateral inscribed in a circle, shew that the area of the triangle* EFG *is to that of the quadrilateral in the ratio* $a^2b^2c^2d^2 : (a^2b^2 \sim c^2d^2)(a^2d^2 \sim b^2c^2)$.

(6) *Prove that the area of a quadrilateral in which a circle can be inscribed is* $\sqrt{abcd}\,\sin\frac{1}{2}(A+C)$; *shew also that* $\sqrt{ad}\,\sin\frac{1}{2}A = \sqrt{bc}\,\sin\frac{1}{2}C$.

(7) *With four given straight lines, three distinct quadrilaterals can be constructed, each of which is inscribable in a circle; their areas are equal; the six diagonals which intersect within the circle are equal in pairs; and if* α, β, γ *be the lengths of these lines,* S *the common area, and* R *the radius of the circle, shew that* $R = \alpha\beta\gamma/4S$.

(8) *The difference of the areas of the triangles whose bases are the sides* b, d *of a quadrilateral, and whose vertices coincide with the intersection of the diagonals, is* $\frac{1}{4}\sqrt{4a^2c^2 - (x^2+y^2-b^2-d^2)^2}$.

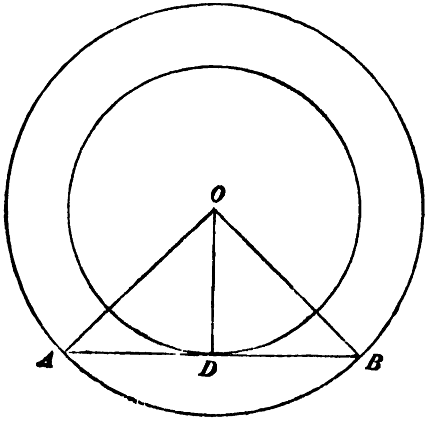
(9) *If a quadrilateral be such that all rectangles described about it are similar, shew that* $a^2+c^2=b^2+d^2$.

(10) *A quadrilateral is such that one circle can be described about it, and another inscribed in it; shew that the radius of the latter is* $\dfrac{2\sqrt{abcd}}{a+b+c+d}$.

(11) *If the diagonals of a quadrilateral intersect in* O, *shew that area* AOB . *area* ABCD = *area* ABC . *area* ABD.

Properties of regular polygons.

168. Let O be the centre of the circles circumscribed about and inscribed in a regular polygon of n sides. Let R, r be the radii of the former and the latter circles, and let a be the length of a side of the polygon.



If AB be a side of the polygon, and D its point of contact with the inscribed circle, the angle AOB is $2\pi/n$, and the angle AOD is π/n; we have

$$a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n}\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

thus the radii of the circles are determined, when the side a is given. The area of the triangle OAB is

$$\frac{1}{2}R^2\sin\frac{2\pi}{n},\quad\text{or}\quad\frac{1}{2}ar,\quad\text{or}\quad r^2\tan\frac{\pi}{n},$$

hence the area of the polygon is

$$\frac{1}{2}nR^2\sin\frac{2\pi}{n}\quad\text{or}\quad nr^2\tan\frac{\pi}{n}\quad\ldots\ldots\ldots\ldots\ldots(29).$$

It should be observed that the problem of inscribing or circumscribing a regular polygon of n sides in, or about a circle, is reduced to the determination of the circular functions of the angle π/n.

169. EXAMPLES.

(1) *Circles are described on the sides* a, b, c *of a triangle as diameters, prove that the diameter* D *of a circle which touches the three externally is such that*

$$\sqrt{\frac{D}{s-a}-1}+\sqrt{\frac{D}{s-b}-1}+\sqrt{\frac{D}{s-c}-1}=\sqrt{\frac{s}{D-s}}.$$

If D, E, F are the middle points of the sides of the given triangle, and O is the centre of circle whose diameter is D, we have

$$OD=\tfrac{1}{2}(D-a),\quad OE=\tfrac{1}{2}(D-b),\quad OF=\tfrac{1}{2}(D-c):$$

also $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$ are the sides of the triangle DEF, thus expressing the areas of the triangles in the relation $\Delta OEF+\Delta OFD+\Delta ODE=\Delta DEF$, in terms of the sides, we obtain the required relation.

(2) *From a point* P, *perpendiculars* PL, PM, PN *are drawn to the sides of a triangle* ABC; *shew that the area of the triangle* LMN *is*

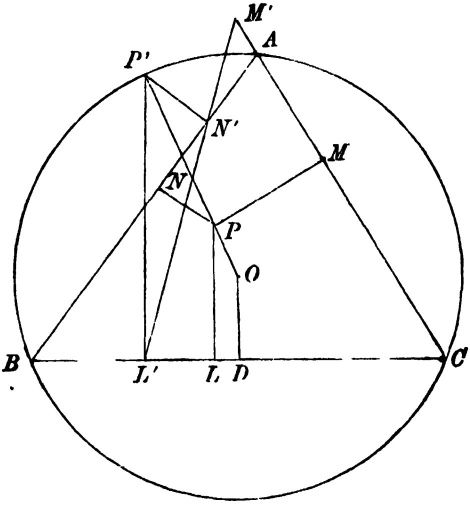
$$\tfrac{1}{4}(R^2-d^2)\sin A\sin B\sin C,$$

where d *is the distance of* P *from the centre of the circum-circle.*

Produce OP to meet the circum-circle in P', and let $P'L'$, $P'M'$, $P'N'$ be drawn perpendicular to the sides, their feet lie on a straight line called the pedal line of P' with respect to the triangle. The perpendicular from a point on the side of a triangle is reckoned as positive or negative according as the point is on the same side or the opposite side of that side as the opposite angle of the triangle.

We have $\frac{PL-OD}{P'L'-OD}=\frac{OP}{OP'}=\frac{d}{R}$, hence $PL=(R-d)\cos A+\frac{d}{R}P'L'$, with similar expressions for PM, PN; now

$$2\Delta LMN=PM\,.\,PN\sin A+PN\,.\,PL\sin B+PL\,.\,PM\sin C$$
$$=(R-d)^2\Sigma\sin A\cos B\cos C+\frac{d^2}{R^2}\Sigma P'M'\,.\,P'N'\sin A$$
$$+\frac{d}{R}(R-d)\Sigma P'L'\sin A;$$



also $\frac{1}{2}\Sigma P'M'\,.\,P'N'\sin A$ is the area of the triangle $L'M'N'$, which is zero, and

$$\Sigma P'L'\sin A=\frac{1}{2R}\Sigma a\,.\,P'L'=\frac{1}{R}\Sigma\Delta P'BC=\frac{1}{R}\Delta ABC,$$

and $$\Sigma\sin A\cos B\cos C=\sin A\sin B\sin C;$$

hence $$2\Delta LMN=(R-d)^2\sin A\sin B\sin C+2d(R-d)\sin A\sin B\sin C$$
$$=(R^2-d^2)\sin A\sin B\sin C.$$

(3) *If* A, B, C *be any three fixed points, and* P *any point on a circle whose centre is* O, *shew that* $AP^2.\Delta BOC+BP^2.\Delta COA+CP^2.\Delta AOB$ *is constant for all positions of* P *on the circle.*

Denote the angles BOC, COA, AOB by α, β, γ, then $\alpha+\beta+\gamma=2\pi$, and let the angle POA be θ. We have $AP^2=OP^2+OA^2-2OA\,.\,OP\cos\theta$, and similar expressions for BP^2, CP^2, hence the expression above is equal to

$$OP^2.\Delta ABC+\Sigma OA^2.\Delta BOC-2OP\Sigma OA\,.\,\Delta BOC\cos\theta;$$

the first two terms in this expression are independent of the position of P on the circle, and the coefficient of $2OP$ in the last term is

$$\tfrac{1}{2}OA\,.\,OB\,.\,OC\,\{\cos\theta\sin\alpha+\cos(\theta+\gamma)\sin\beta+\cos(\beta-\theta)\sin\gamma\}$$

or
$$\tfrac{1}{2}OA\,.\,OB\,.\,OC\cos\theta\,(\sin\alpha+\sin\beta\cos\gamma+\cos\beta\sin\gamma)$$

which is zero; thus the theorem is proved.

Particular cases of this theorem are the following:

(*a*) $PA^2\sin 2A+PB^2\sin 2B+PC^2\sin 2C$ is constant if P lies on the circum-circle;

(*b*) $PA^2\sin A+PB^2\sin B+PC^2\sin C$ is constant if P lies on the in-circle.

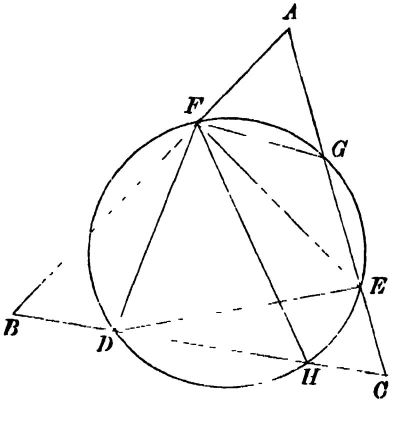
(*c*) $PA^2\sin A\cos(B-C)+PB^2\sin B\cos(C-A)+PC^2\sin C\cos(A-B)$ is constant if P lies on the nine-point circle.

(4) *Shew that the length of the side of the least equilateral triangle that can be drawn with its angular points on the sides of a given triangle* ABC *is*

$$\frac{2\Delta\sqrt{2}}{\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}},$$

where Δ *is the area of* ABC.

Let DEF be such an equilateral triangle, and let the circle round DEF cut BC and AC in H and G respectively; the angles FGA, FHB are each 60° thus FG, FH are in fixed directions; also the angle HFG is $120°-C$.



We have, if AF be denoted by x,

$$FG=x\sin A/\sin 60°,\quad FH=(c-x)\sin B/\sin 60°,$$

hence

$$HG^2=\operatorname{cosec}^2 60°\,\{x^2\sin^2 A+(c-x)^2\sin^2 B-2x(c-x)\sin A\sin B\cos(120°-C)\}.$$

Now the radius of the circle is $HG/2\sin(120°-C)$, hence the circle is least when HG is least. The least value of a quadratic expression $\lambda x^2+2\mu x+\nu$, in which λ is positive, is $\nu-\frac{\mu^2}{\lambda}$, for $\lambda x^2+2\mu x+\nu$ may be written in the form $\lambda\left(x+\frac{\mu}{\lambda}\right)^2+\nu-\frac{\mu^2}{\lambda}$. We find therefore for the least value of $HG\sin 60°$,

$$\left\{c^2\sin^2 B-\frac{(c\sin^2 B+c\sin A\sin B\cos\overline{120°-C})^2}{\sin^2 A+\sin^2 B+2\sin A\sin B\cos(120°-C)}\right\}^{\frac{1}{2}},$$

which is equal to

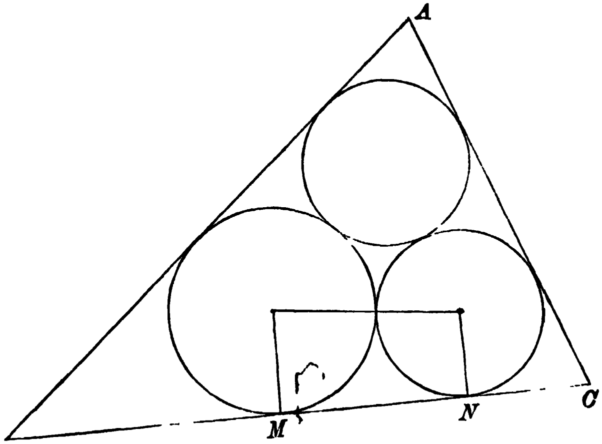
$$\frac{c\sin A\sin B\sin(120°-C)}{\{\sin^2 A+\sin^2 B+2\sin A\sin B\cos(120°-C)\}^{\frac{1}{2}}},$$

or

$$\frac{\sqrt{2}\,c^2\sin A\sin B\sin(120°-C)}{\sin C\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}}.$$

Now the side of the equilateral triangle is $HG\sin 60°/\sin(120°-C)$, thus the least value of the side is $\dfrac{2\Delta\sqrt{2}}{\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}}$.

(5) *Describe three circles mutually in contact, each of which touches two sides of a given triangle.*



Let ρ_1, ρ_2, ρ_3 be the radii of the circles, then $MN=2\sqrt{\rho_2\rho_3}$,

hence $$a=BM+CN+MN=\rho_2\cot\tfrac{1}{2}B+\rho_3\cot\tfrac{1}{2}C+2\sqrt{\rho_2\rho_3},$$

with similar equations for b and c.

Let $$x^2=\rho_1\cot\tfrac{1}{2}A,\quad y^2=\rho_2\cot\tfrac{1}{2}B,\quad z^2=\rho_3\cot\tfrac{1}{2}C,$$

$\sqrt{\tan\frac{1}{2}B\tan\frac{1}{2}C}=-\cos\alpha$, $\sqrt{\tan\frac{1}{2}C\tan\frac{1}{2}A}=-\cos\beta$, $\sqrt{\tan\frac{1}{2}A\tan\frac{1}{2}B}=-\cos\gamma$; we find $\sin^2\alpha=1-\tan\frac{1}{2}B\tan\frac{1}{2}C=a/s$, and similarly $\sin^2\beta=b/s$, $\sin^2\gamma=c/s$, hence we have the equations

$$\frac{y^2+z^2-2yz\cos\alpha}{\sin^2\alpha}=\frac{z^2+x^2-2zx\cos\beta}{\sin^2\beta}=\frac{x^2+y^2-2xy\cos\gamma}{\sin^2\gamma}=s;$$

these have been considered in Art. 68, Ex. (12); adopting the first solution there found, we have

$$x=\sqrt{s}\cos(\sigma-\alpha),\quad y=\sqrt{s}\cos(\sigma-\beta),\quad z=\sqrt{s}\cos(\sigma-\gamma),$$

where

$$2\sigma=\alpha+\beta+\gamma,$$

hence

$$\rho_1=s\tan\tfrac{1}{2}A\cos^2(\sigma-\alpha),\quad \rho_2=s\tan\tfrac{1}{2}B\cos^2(\sigma-\beta),\quad \rho_3=s\tan\tfrac{1}{2}C\cos^2(\sigma-\gamma)$$

are the required radii of the circles. The other solutions give the radii of three sets of circles which are such that two in each set touch two sides of the triangle produced; of one such set, the radii are

$$s\tan\tfrac{1}{2}A\cos^2 s,\quad s\tan\tfrac{1}{2}B\cos^2(s-\gamma),\quad s\tan\tfrac{1}{2}C\cos^2(s-\beta).$$

There are altogether eight sets of circles which satisfy the conditions of the problem.

This solution is founded on that of Lechmütz given in the *Nouvelles Annales*, Vol. v. A geometrical solution of this problem, which is known as "Malfatti's Problem," will be found in Casey's *Sequel to Euclid.* A history of the problem will be found in the *Bulletin de l'Académie Royale de Belgique* for 1874, by M. Simons.

EXAMPLES ON CHAPTER XII.

1. If θ be the angle between the diagonals of a parallelogram whose sides a, b are inclined at an angle α to each other, shew that $\tan\theta=\dfrac{2ab\sin\alpha}{a^2-b^2}$.

2. If α, β, γ be the distances, from the angular points of a triangle, to the points of contact of the inscribed circle with the sides, shew that

$$r=\left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}.$$

3. The area of a regular inscribed polygon is to that of the circumscribed polygon, of the same number of sides, as 3 : 4; find the number of sides.

4. From each angle of a parallelogram a line is drawn making the same angle, towards the same parts, with an adjacent side, taken always in the same order; shew that these lines will form another parallelogram similar to the original one, if $a^2\sim b^2=2ab\cos B$, where a, b are the sides, and B is an angle of the parallelogram.

5. The straight lines which bisect the angles A, C of a triangle meet the circumference of the circum-circle in the points α, γ; shew that the straight line $\alpha\gamma$ is divided by CB, BA into three parts which are in the ratio

$$\sin^2\tfrac{1}{2}A : 2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C : \sin^2\tfrac{1}{2}C.$$

6. If I be the centre of the in-circle of a triangle, Ia, Ib, Ic perpendiculars on the sides, ρ_1, ρ_2, ρ_3 the radii of circles inscribed in the quadrilaterals $AbIc$, $BcIa$, $CaIb$, prove that

$$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3}=\frac{a+b+c}{2r}.$$

7. Prove that the line joining the centres of the circum-circle and the in-circle of a triangle makes with BC an angle $\cot^{-1}\left(\frac{\sin B \sim \sin C}{\cos B+\cos C-1}\right)$.

8. If, in a triangle, the feet of the perpendiculars from two angles, on the opposite sides, be equally distant from the middle points of those sides, shew that the other angle is 60°, or 120°, or else the triangle is isosceles.

9. If ABC be a triangle having a right-angle at C, and AE, BD drawn perpendicularly to AB meet BC, AC produced in E, D respectively, prove that $\tan CED=\tan^3 BAC$, and $\Delta\, ECD=\Delta\, ACB$

10. If a point be taken within an equilateral triangle, such that its distances from the angular points are proportional to the sides a, b, c of another triangle, shew that the angles between these distances will be

$$\tfrac{1}{3}\pi+A,\quad \tfrac{1}{3}\pi+B,\quad \tfrac{1}{3}\pi+C.$$

11. The points of contact of each of the four circles touching the three sides of a triangle are joined; prove that, if the area of the triangle thus formed from the inscribed circle be subtracted from the sum of the areas of those formed from the escribed circles, the remainder will be double of the area of the original triangle.

12. If $ABCD$ is a parallelogram and P is any point within it, prove that $\Delta\, APC \,.\, \cot APC - \Delta\, BPD \,.\, \cot BPD$ is independent of the position of P.

13. Three circles touching each other externally are all touched by a fourth circle including them all. If a, b, c be the radii of the three internal circles, and α, β, γ the distances of their centres from that of the external circle respectively, prove that

$$2\left(\frac{\beta\gamma}{bc}+\frac{\gamma\alpha}{ca}+\frac{\alpha\beta}{ab}\right)=4+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}.$$

14. P, Q, R are points in the sides BC, CA, AB of a triangle, such that $\frac{BP}{PC}=\frac{CQ}{QA}=\frac{AR}{BR}$; shew that $AP^2+BQ^2+CR^2$ is least, when P, Q, R bisect the sides.

15. On the sides a, b, c of a triangle are described segments of circles external to the triangle, containing angles α, β, γ respectively, where $\alpha+\beta+\gamma=\pi$, and a triangle is formed by joining the centres of these circles; shew that the angles of this triangle are α, β, γ.

16. Through the middle points of the sides of a triangle, straight lines are drawn perpendicular to the bisectors of the opposite angles, and form another triangle; prove that its area is a quarter of the rectangle contained by the perimeter of the former triangle and the radius of the circle described about it.

17. P is a point in the plane of a triangle ABC, and L, M, N are the feet of the perpendiculars from P on the sides; prove that if $MN+NL+LM$ be constant and equal to l, the least value of

$$PA^2+PB^2+PC^2 \text{ is } l^2/(\sin^2 A+\sin^2 B+\sin^2 C).$$

18. Lines $B'C'$, $C'A'$, $A'B'$ are drawn parallel to the sides BC, CA, AB of a triangle, at distances r_1, r_2, r_3 respectively; find the area of the triangle $A'B'C'$.

If eight triangles be so formed, the mean of their perimeters is equal to the perimeter of the triangle ABC, but the mean of their areas exceeds its area by

$$(a^2r_1^2+b^2r_2^2+c^2r_3^2)/4\Delta$$

19. On the sides of a scalene triangle ABC, as bases, similar isosceles triangles are described, either all externally or all internally, and their vertices are joined so as to form a new triangle $A'B'C'$; prove that if $A'B'C'$ be equilateral, the angles at the base of the isosceles triangles are each 30°; and that if the triangle $A'B'C'$ be similar to ABC, the angles are each

$$\tan^{-1}\frac{4\Delta}{a^2+b^2+c^2},$$

where Δ is the area of ABC.

20. A straight line cuts three concentric circles in A, B, C, and passes at a distance p from their centre; shew that the area of the triangle formed by the tangents at A, B, C is $\dfrac{BC\,.\,CA\,.\,AB}{2p}$.

21. If N is the centre of the nine-point circle of a triangle ABC, and D, E, F are the middle points of the sides, prove that

$$BC\cos NDC+CA\cos NEA+AB\cos NFB=0.$$

22. On the side BA of a triangle is measured BD equal to AC; BC and AD are bisected in E and F; E and F are joined; shew that the radius of the circle round BEF is $\frac{1}{4}BC\operatorname{cosec}\frac{1}{2}A$.

23. If A', B', C' be any points on the sides of the triangle ABC, prove that

$$AB'\,.\,BC'\,.\,CA'+B'C\,.\,C'A\,.\,A'B=4R\,.\,\Delta A'B'C'.$$

24. If x, y, z denote the distances of the centre of the in-circle of a triangle from the angular points, shew that

$$a^4x^4+b^4y^4+c^4z^4+(a+b+c)^2x^2y^2z^2=2(b^2c^2y^2z^2+c^2a^2z^2x^2+a^2b^2x^2y^2).$$

25. D, E, F are the points where the bisectors of the angles of the triangle ABC meet the opposite sides; if x, y, z are the perpendiculars

drawn from A, B, C, respectively, to the opposite sides of DEF, p_1, p_2, p_3 those drawn from A, B, C, respectively, to the opposite sides of ABC, prove that

$$\frac{p_1^2}{x^2}+\frac{p_2^2}{y^2}+\frac{p_3^2}{z^2}=11+8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

26. Shew that the distances of the orthocentre of a triangle from the angular points are the roots of the equation

$$x^3-2(R+r)x^2+(r^2-4R^2+s^2)x-2R\{s^2-(r+2R)^2\}=0.$$

27. If each side of a triangle bears to the perimeter a ratio less than 2 : 5, a triangle can be formed, having its sides equal to the radii of the escribed circles.

28. ABC is a triangle inscribed in a circle, and from D, the middle point of BC, a line is drawn at right angles to BC, meeting the circumference in E and F; AE, AF are joined. If triangles be described in the same way by bisecting AB, AC, shew that the areas of the three triangles thus formed are as

$$\sin(B-C):\sin(C-A):\sin(A-B).$$

29. Three circles, whose radii are a, b, c, touch each other externally; prove that the radii of the two circles which can be drawn to touch the three are

$$\frac{abc}{(bc+ca+ab)\pm2\sqrt{abc(a+b+c)}}.$$

30. ABC is a triangle; on its sides equilateral triangles $A'BC$, $B'CA$, $C'AB$ are described without the triangle; prove that (1) AA', BB', CC' meet in a point O, (2) $OA'=OB+OC$,

$$(3)\ \Delta A'B'C'=\tfrac{5}{2}\Delta ABC+\frac{\sqrt{3}}{8}(BC^2+CA^2+AB^2).$$

31. A', B' are the middle points of the sides a, b of a triangle; D, E are the feet of the perpendiculars from A, B on the opposite sides; $A'D$, $B'E$ are bisected in P, Q; prove that $PQ=\frac{1}{4}\sqrt{a^2+b^2-2ab\cos3C}$.

32. The perpendiculars from the angular points of an acute-angled triangle meet in P, and PA, PB, PC are taken for sides of a new triangle. Find the condition that this is possible, and if it is, and α, β, γ are the angles of the new triangle, prove that

$$1+\frac{\cos\alpha}{\cos A}+\frac{\cos\beta}{\cos B}+\frac{\cos\gamma}{\cos C}=\tfrac{1}{2}\sec A\sec B\sec C.$$

33. Two points A, B are taken within a circle of radius r, whose centre is C. Prove that the diameters of the circles which can be drawn through A and B to touch the given circle are the roots of the equation

$$x^2(r^2c^2-a^2b^2\sin^2C)-2xrc^2(r^2-ab\cos C)+c^2(r^4-2r^2ab\cos C+a^2b^2)=0,$$

where the symbols refer to the parts of the triangle ABC.

34. If a triangle be cut out in paper, and doubled over so that the crease passes through the centre of the circumscribed circle and one of the angles A, shew that the area of the doubled portion is

$$\tfrac{1}{2}b^2\sin^2 C\cos C\operatorname{cosec}(2C-B)\sec(C-B), \quad \text{where} \quad C>B.$$

35. From the feet of the perpendiculars from the angular points A, B, C of a triangle, on the opposite sides, perpendiculars are drawn to the adjacent sides; shew that the feet of these six perpendiculars lie on a circle whose radius is

$$R\,(\cos^2 A\cos^2 B\cos^2 C+\sin^2 A\sin^2 B\sin^2 C)^{\frac{1}{2}}.$$

36. Prove that if P be a point from which tangents to the three escribed circles of the triangle ABC are equal, the distance of P from the side BC will be

$$\tfrac{1}{2}(b+c)\sec\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

37. If x, y, z be the sides of the squares inscribed in the triangle ABC, on the sides BC, CA, AB, shew that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{r}$.

38. AA', BB', CC' are the perpendiculars from A, B, C on the opposite sides of the triangle ABC; O_1, O_2, O_3 are the orthocentres of the triangles $AB'C'$, $BC'A'$, $CA'B'$. Prove (1) that the triangles $O_1O_2O_3$, $A'B'C'$ are equal, and (2) that $2r_1R_1^2=R_aR_bR_c$, where R_a, R_b, R_c are the radii of the circles $O_2A'O_3$, $O_3B'O_1$, $O_1C'O_2$, and r_1 is the radius of the circle inscribed in $A'B'C'$, and R_1 of the circle about $A'B'C'$.

39. If x, y, z are the distances of the centres of the escribed circles of a triangle, from the centre of the in-circle, and d is the diameter of the circum-circle, shew that

$$xyz+d\,(x^2+y^2+z^2)=4d^3.$$

40. The lines joining the centre of the in-circle of a triangle, to the angular points, meet that circle in A_1, B_1, C_1; prove that the area of the triangle $A_1B_1C_1$ is $\frac{1}{2}r^2(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)$.

41. If each side of a triangle be increased by the same small quantity x, shew that the area is increased by $Rx\,(\cos A+\cos B+\cos C)$, nearly.

42. AA', BB', CC' are diameters of a circle, D, E, F are the feet of the perpendiculars from A', B', C' on BC, CA, AB respectively; prove that AD, BE, CF meet in a point, and that the areas ABC, DEF are in the ratio

$$1 : 2\cos A\cos B\cos C.$$

43. If ID, IE, IF are drawn from the in-centre I of a triangle, perpendicular to the sides, find the radii of the circles inscribed in $IEAF$, $IFBD$, $IDCE$; if they are denoted by ρ_1, ρ_2, ρ_3 respectively, shew that

$$r-2\rho_1)(r-2\rho_2)(r-2\rho_3)=r^3-4\rho_1\rho_2\rho_3.$$

44. Shew that the radii of the circle which touches externally each of three given circles, of radii a, b, c which touch each other externally, is given by

$$\sqrt{Rbc(b+c+R)}+\sqrt{Rca(c+a+R)}+\sqrt{Rab(a+b+R)}=\sqrt{abc(a+b+c)}.$$

45. Perpendiculars AA_1, BB_1, CC_1 to the plane of a triangle ABC are erected at its angular points, and their respective lengths are a, b, c; shew that if Δ and Δ_1 be the areas of ABC and $A_1B_1C_1$, then

$$\begin{aligned}\Delta_1{}^2-\Delta^2&=\tfrac{1}{4}\{a^2(x-y)(x-z)+b^2(y-z)(y-x)+c^2(z-x)(z-y)\}\\&=\tfrac{1}{4}\{a_1{}^2(x-y)(x-z)+b_1{}^2(y-z)(y-x)+c_1{}^2(z-x)(z-y)\}.\end{aligned}$$

46. Three circles are described, each touching two sides of a triangle, and also the inscribed circle. Shew that the area of the triangle having their centres for angular points bears to the area of the given triangle the ratio

$$\begin{aligned}&4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C(\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C)\\&\qquad:\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C).\end{aligned}$$

47 If the lines bisecting the angles of a triangle meet the opposite sides in D, E, F, prove that the area of the triangle DEF is

$$2r^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C/\cos\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A-B),$$

and that

$$(a+b)^2(a+c)^2EF^2+(b+c)^2(b+a)^2FD^2+(c+a)^2(c+b)^2DE^2=16\Delta^2R(11R+2r),$$

where Δ is the area of ABC.

48. O is the centre of the circum-circle of a triangle, K is the orthocentre, and OK meets the circle in P and P', and the pedal lines of P and P' in Q and Q'; prove that $OQ\,.\,OQ'=2R^2\cos A\cos B\cos C$.

49. N is the centre of the nine-point circle of a triangle; D, E are the middle points of CB and CA; prove that the area of the quadrilateral $NDCE$ is $\frac{1}{2}\rho^2(\sin 2A+\sin 2B+2\sin 2C)$, where ρ is the radius of the nine-point circle.

50. A triangle is formed by joining the centres of the escribed circles, a third from this, and so on; shew that the sides of the nth triangle are

$$a\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{\pi-A}{2^2}\operatorname{cosec}\frac{3\pi+A}{2^3}\ldots\ldots\operatorname{cosec}\frac{(2^{n-2}-1)\pi+(-1)^{n-2}A}{2^{n-1}},$$

and similar expressions.

51. If N is the centre of the nine-point circle of ABC, and AN meets BC in D, shew that

$$DN:DA::\cos(B-C):4\sin B\sin C,$$

and that the area of BNC is $\frac{1}{2}R^2\sin A\cos(B-C)$.

52. Shew that the radius of the circle which touches the three circles DCE, EAF, FBD, where D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides, is

$$\frac{2R\sin A\sin B\sin C\cos A\cos B\cos C(\sin A+\sin B+\sin C)}{\sin^2 A\sin^2 B\sin^2 C-\Sigma\sin^2 A\cos^2 A+2\cos A\cos B\cos C\,\Sigma\sin B\sin C}\cdot$$

53 If from any point O, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB of a triangle, prove that $\cot ADC + \cot BEA + \cot CFB = 0$.

54. If b, c, B are given, and there are two triangles with these given parts, shew that their inscribed circles touch, if

$$c^2(\cos^2 B + 2\cos B - 3) + 2bc(1 - \cos B) + b^2 = 0.$$

55. If t_1, t_2, t_3 be the lengths of the tangents drawn from the centres of the escribed circles of a triangle to the nine-point circle, shew that

$$\frac{t_1^2}{r_1} + \frac{t_2^2}{r_2} + \frac{t_3^2}{r_3} = r + 7R, \text{ and } \frac{t_1^2 - t_2^2}{r_1 - r_2} + \frac{t_2^2 - t_3^2}{r_2 - r_3} + \frac{t_3^2 - t_1^2}{r_3 - r_1} = 2r + 11R.$$

56. Prove that the sum of the squares of the distances of the centre of the nine-point circle of a triangle, from the angular points, is

$$R^2\left(\tfrac{11}{4} + 2\cos A \cos B \cos C\right).$$

57. Four similar triangles are described about a given circle, and their areas are Δ, Δ_1, Δ_2, Δ_3, shew that

(a) an angle of the triangles is $2\cot^{-2}\left(\dfrac{\Delta\Delta_1}{\Delta_2\Delta_3}\right)^{\frac{1}{4}}$,

(b) $\Delta^{\frac{1}{2}} = \Delta_1^{\frac{1}{2}} + \Delta_2^{\frac{1}{2}} + \Delta_3^{\frac{1}{2}}$,

(c) the radius of the circle is $(\Delta\Delta_1\Delta_2\Delta_3)^{\frac{1}{8}}$.

58. Through the angles A, B, C of a triangle, straight lines are drawn making angles θ, ϕ, ψ with the opposite sides of the triangle, in the same sense. Prove that the diameter of the circle circumscribing the triangle formed by these lines is

$$R \cdot \frac{\sin(2A + \phi - \psi)\cos\theta + \sin(2B + \psi - \theta)\cos\phi + \sin(2C + \theta - \phi)\cos\psi}{\sin(A + \phi - \psi)\sin(B + \psi - \theta)\sin(C + \theta - \phi)}.$$

59. The sides of a triangle subtend angles α, β, γ at a point O; prove that

(1) $\cos\frac{1}{4}\alpha + \cos\frac{1}{4}\beta + \cos\frac{1}{4}\gamma = 4\cos\frac{1}{4}(\beta + \gamma)\cos\frac{1}{4}(\gamma + \alpha)\cos\frac{1}{4}(\alpha + \beta)$,

(2) $OA = \dfrac{bc\sin(\alpha - A)}{\sqrt{bc\sin\alpha\sin(\alpha - A) + ca\sin\beta\sin(\beta - B) + ab\sin\gamma\sin(\gamma - C)}}$.

60. If d_1, d_2, d_3 be the distances of any point in the plane of an equilateral triangle whose side is a, from the angular points, prove that

$$d_2^2 d_3^2 + d_3^2 d_1^2 + d_1^2 d_2^2 + a^2(d_1^2 + d_2^2 + d_3^2) = a^4 + d_1^4 + d_2^4 + d_3^4.$$

Hence shew that the sum of two equilateral triangles, each of which has its vertices at three given distances from a fixed point, is equal to the sum of the equilateral triangles described on the distances.

61. If P be any point within a triangle ABC, and O_1, O_2, O_3 are the circum-centres of the triangles BPC, CPA, APB respectively, then if ρ be the circum-radius of $O_1O_2O_3$, shew that

$$4\rho\sin\theta\sin\phi\sin\psi = x\sin\theta + y\sin\phi + z\sin\psi,$$

where x, y, z are the lengths PA, PB, PC, and θ, ϕ, ψ are the angles BPC, CPA, APB.

62. If a, b, c be the radii of three circles touching each other externally, and r_1, r_2 be the radii of the two circles that can be drawn to touch these three, shew that $\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{a}+\frac{2}{b}+\frac{2}{c}$.

63. If the bisectors of the angles B, C, of a triangle, meet the opposite sides in E, F, prove that EF makes with BC an angle

$$\tan^{-1}\frac{(b-c)\sin A}{(a+b)\cos C+(a+c)\cos B}.$$

64. If I be the centre of the circle inscribed in ABC, I_1 that of the circle inscribed in IBC, I_2 that of the circle inscribed in I_1BC, and so on; shew that as n indefinitely increases, I_nI_{n-1} divides BC in the ratio of the measures of the angles C and B.

65. Points D, E, F are taken on the sides BC, CA, AB of a triangle, and through D, E, F are drawn straight lines $B'C'$, $C'A'$, $A'B'$, equally inclined to BC, CA, AB respectively, so as to form a triangle $A'B'C'$ similar to ABC. Prove that the radius of the circumscribed circle of $A'B'C'$ is

$$(EF\cos\alpha+FD\cos\beta+DE\cos\gamma)/4\sin A\sin B\sin C,$$

where α, β, γ are the inclinations of AA', BB', CC' to BC, CA, AB respectively.

66. If P be a point on the circum-circle whose pedal line passes through the centroid, and if the line joining P to the orthocentre cuts the pedal line at right angles, prove that

$$PA^2+PB^2+PC^2=4R^2(1-2\cos A\cos B\cos C).$$

67. D is a point in the side BC of a triangle; if the circles inscribed in the triangles ABD, ACD touch AD in the same point, prove that D is the point of contact of the in-circle of ABC with BC; but if the radii of the circles be equal, then

$$CD : BD :: \operatorname{cosec} D+\operatorname{cosec} C : \operatorname{cosec} D+\operatorname{cosec} B.$$

68. From a point within a circle of radius r, three radii vectores of lengths r_1, r_2, r_3 are drawn to the circle, and the angle contained by any pair is $2\pi/3$; shew that

$$3r^2(r_2r_3+r_3r_1+r_1r_2)^2=(r_2^2+r_2r_3+r_3^2)(r_3^2+r_3r_1+r_1^2)(r_1^2+r_1r_2+r_2^2),$$

and that the distance of the point from which the radii are drawn, from the centre of the circle, is d, where

$$(r^2-d^2)(r_2r_3+r_3r_1+r_1r_2)=r_1r_2r_3(r_1+r_2+r_3).$$

69. Circles are inscribed in the triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$, where D_1, E_1, F_1 are the points of contact of the circle escribed to the side BC; shew that if ρ_1, ρ_2, ρ_3 be the radii of these circles

$$\frac{1}{\rho_1}:\frac{1}{\rho_2}:\frac{1}{\rho_3}=1-\tan\tfrac{1}{4}A:1-\tan\tfrac{1}{4}B:1-\tan\tfrac{1}{4}C.$$

70. In a triangle ABC, A', B', C' are the centres of the circles described each touching two sides and the inscribed circle; shew that the area of the triangle $A'B'C'$ is

$$\tan \tfrac{1}{4}(\pi - A) \tan \tfrac{1}{4}(\pi - B) \tan \tfrac{1}{4}(\pi - C)$$
$$\{\operatorname{cosec} \tfrac{1}{4}(\pi - A) \operatorname{cosec} \tfrac{1}{4}(\pi - B) \operatorname{cosec} \tfrac{1}{4}(\pi - C) + 4\} r^2.$$

71. The three tangents to the in-circle of a triangle which are parallel to the sides are drawn; shew that the radii of the circles inscribed in the three triangles so cut off from the corners are given by the equation

$$s^2x^3 - rs^2x^2 - \tfrac{1}{4}r^2(a^2+b^2+c^2-2bc-2ca-2ab)\,x - r^5 = 0.$$

72. The perpendiculars from the angular points of a triangle on the straight line joining the orthocentre and the centre of the in-circle are p, q, r; prove that

$$\frac{p \sin A}{\sec B - \sec C} = \frac{q \sin B}{\sec C - \sec A} = \frac{r \sin C}{\sec A - \sec B},$$

a convention being made as to the signs of p, q, r.

73. A point is taken within an equilateral triangle, and its distances from the angular points are a, β, γ. The internal bisectors of the angles between (β, γ), (γ, a), (a, β) meet the corresponding sides of the triangle in P, Q, R respectively; shew that the area of PQR is to that of the equilateral triangle in the ratio

$$2a\beta\gamma : (\beta+\gamma)(\gamma+a)(a+\beta).$$

74 If l, m, n are the distances of any point in the plane of a triangle ABC, from its angular points, and d the distance from the circum-centre, prove that

$$l^2 \sin 2A + m^2 \sin 2B + n^2 \sin 2C = 4(R^2+d^2) \sin A \sin B \sin C.$$

75. If G is the centroid of a triangle, shew that

$$\cot GAB + \cot GBC + \cot GCA = 3 \cot \omega = \cot ABG + \cot BCG + \cot CAG,$$

and $$\cot AGB + \cot BGC + \cot CGA + \cot \omega = 0,$$

where $$\cot \omega = \cot A + \cot B + \cot C.$$

Also if K be the symmedian point, that is a point in the triangle, such that the angles KAC, GAB are equal, and two similar relations, then

$$\cot AKB + \cot BKC + \cot CKA + \tfrac{1}{2} \cot \omega + \tfrac{3}{2} \tan \omega = 0.$$

76. Each of three circles, within the area of a triangle, touches the other two, touching also two sides of the triangle; if a be the distance between the points of contact of one of the sides, and β, γ be like distances on the other two sides, prove that the area of the triangle of which the centres of the circles are angular points is $\tfrac{1}{4}(\beta^2\gamma^2 + \gamma^2a^2 + a^2\beta^2)^{\frac{1}{2}}$.

77. If a, b, c, d be the perpendiculars from the angles of a quadrilateral upon the diagonals d_1, d_2, shew that the sine of the angle between the diagonals is equal to $\left\{\dfrac{(a+c)(b+d)}{d_1d_2}\right\}^{\frac{1}{2}}$.

78. If $ABCD$ be a quadrilateral, prove, in any manner, that the line joining the intersection of the bisectors of the angles A and C with the intersection of the angles B and D makes with AD an angle equal to

$$\tan^{-1}\left\{\frac{\sin A-\sin D+\sin(A+B)}{1+\cos A+\cos D+\cos(A+B)}\right\}.$$

79. $ABCDE$ is a plane pentagon; having given that the areas of the triangles EAB, ABC, BCD, CDE, DEA are equal to a, b, c, d, e respectively, shew that the area A of the polygon may be found from the equation

$$A^2-(a+b+c+d+e)A+(ab+bc+cd+de+ea)=0.$$

80. Shew that if a quadrilateral whose sides, taken in order, are a, b, c, d be such that a circle can be inscribed in it, the circle is the greatest when the quadrilateral can be inscribed in a circle, and that then the square on the radius of the inscribed circle is $\dfrac{abcd}{(a+c)(b+d)}$.

81. A polygon of $2n$ sides, n of which are equal to a, and n to b, is inscribed in a circle; shew that the radius of the circle is

$$\frac{1}{2}\left(a^2+2ab\cos\frac{\pi}{n}+b^2\right)^{\frac{1}{2}}\operatorname{cosec}\frac{\pi}{n}.$$

82. A quadrilateral whose sides are a, b, c, d can be inscribed in a circle; its external angles are bisected; prove that the diagonals of the quadrilateral formed by these bisecting lines are at right angles, and that the area of this quadrilateral is $\dfrac{1}{2}\dfrac{s^2(ab+cd)(ad+bc)}{(a+c)(b+d)\sqrt{(s-a)(s-b)(s-c)(s-d)}}$,

where $$2s=a+b+c+d.$$

83. A quadrilateral $ABCD$ is inscribed in a circle, and EF is its third diagonal, which is opposite to the vertex A; prove that if the perpendiculars from A on BC, CD meet the circles described on AD, AB respectively as diameters, in P, then $PQ\sin D=EF(\sin^2 A-\sin^2 D)$.

84. The power of two circles with regard to one another, is defined to be the excess of the square of the distance between their centres, over the sum of the squares of the radii. Prove that for a triangle ABC, the power of the inscribed circle, and that escribed circle which is opposite A, is $\frac{1}{2}\{a^2+(b-c)^2\}$, and hence verify that if the escribed circle touches an escribed circle, the triangle must be isosceles.

85. The sides, taken in order, of a pentagon circumscribed to a circle are a, b, c, d, e; prove that its area is a root of the equation

$$x^4-x^2s\{\tfrac{1}{4}\Sigma a^2(b+e-c-d)-\tfrac{1}{4}\Sigma a^3+\tfrac{1}{2}\Sigma acd\}$$
$$+(s-a-e)(s-b-d)(s-c-e)(s-d-a)(s-c-b)s^3=0,$$

where $2s$ is the sum of the sides.

86. If a, b, c, d be the distances of any point on the circumference of a circle of radius r, from four consecutive angular points of an inscribed regular polygon, find the relation between a, b, c, and d, and prove that

$$r^2 = \frac{(ab-cd)(bc-ad)(ca-bd)}{(a+b-c-d)(b+c-a-d)(c+a-b-d)(a+b+c+d)}.$$

87. The perimeter and area of a convex pentagon $ABCDE$, inscribed in a circle, are $2s$ and S, and the sum of the angles at E and B, at A and C, ... are denoted by α, β, ...; shew that

$$s^2(\sin 2\alpha + \ldots\ldots + \sin 2\epsilon) + 2S(\sin\alpha + \ldots\ldots + \sin\epsilon)^2 = 0.$$

88. $ABCD$ is a convex quadrilateral of which the sides touch one circle, while the vertices lie on another; tangents are drawn to the circumscribed circle at A, B, C, D so as to form another convex quadrilateral; prove that the area of the latter is

$$2r^2 \frac{(s\sigma - 2abcd)(abcd)^{\frac{3}{2}}\sigma}{(\sigma-bcd)(\sigma-cda)(\sigma-dab)(\sigma-abc)},$$

where r is the radius of the circle $ABCD$, $2s = a+b+c+d$, and

$$2\sigma = bcd + cda + dab + abc.$$

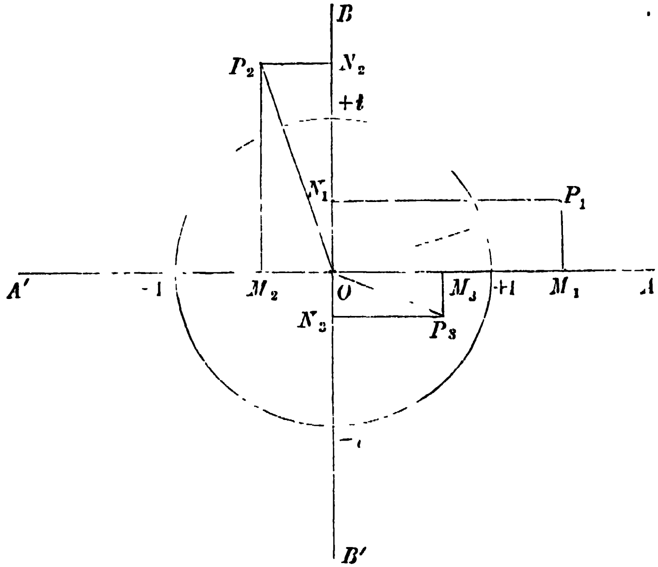
CHAPTER XIII.

COMPLEX NUMBERS.

170. In works on Algebra, numbers of the form $x+iy$, called complex numbers, are considered, and the application to them of the ordinary laws of algebraical operations is justified. We shall, in this Chapter, consider the mode in which such complex numbers may be geometrically represented, and in which the results of additions and multiplications of such numbers may be exhibited. It will appear that circular functions present themselves naturally in this connection, and indeed that such functions must be introduced in order to give conciseness to the results of the multiplication and division of complex numbers.

The geometrical representation of a complex number.

171. A positive or negative real number x is represented geometrically by laying off on a fixed infinite straight line $A'OA$, a length $OM=|x|$, to scale, measured from any specified point O in one direction or the other, according as x is positive or negative; we may then consider that the number x is represented either by the position of the point M, or by the straight line OM. In order to represent a purely imaginary number iy, take a fixed straight line $B'OB$, in any fixed plane containing $A'OA$, perpendicular to the latter line, then measure from O a length $ON=|y|$, in the direction OB or OB', according as y is positive or negative, then we shall consider that the imaginary number iy is represented by the point N, or also by the straight line ON. A circle of radius unity cuts $A'A$ and $B'B$ in the points which represent the numbers ± 1, $\pm i$ respectively. In order to represent the complex number $x+iy$, complete the rectangle $OMPN$, then we shall consider that the point P, or also the straight line OP, represents $x+iy$. We thus suppose that the result of the addition of the two numbers x and iy is represented geometrically by the diagonal of the parallelogram of which the two straight lines OM, ON, which represent x and iy respectively, are sides. In the figure, P_1 represents a number x_1+iy_1 in which both x_1 and y_1 are positive, P_2 a number x_2+iy_2 in which x_2 is negative and y_2 is positive, and P_3 a number x_3+iy_3 in which x_3 is positive and y_3 is negative. $A'OA$ is called the real axis, and $B'OB$ the imaginary axis.



172. Let r denote the absolute length of OP, and θ the angle which OP makes with OA, measured counter-clockwise from OA, then

$$x=r\cos\theta,\quad y=r\sin\theta,\quad \text{and}\quad z=x+iy=r(\cos\theta+i\sin\theta),$$

where
$$r=\sqrt{x^2+y^2},\quad \theta=\tan^{-1}\frac{y}{x}.$$

The essentially positive number $r=\sqrt{x^2+y^2}$ is called the *modulus*, and the angle θ is called the *argument* of the complex number

$x+iy$. A straight line OP measured in any direction from O in the plane is thus capable in virtue of its two qualities of absolute length, and of direction, of completely representing a complex number. The number $x+iy$ may also be represented by any straight line in the plane, drawn parallel to OP, and of equal length, since such a straight line represents both the modulus and the argument of $x+iy$.

173. Suppose a point P to describe a circle with centre O, and any radius r, commencing from A' and moving in the counter-clockwise direction, then the modulus of the complex number represented by P remains constant and equal to r, whilst the argument increases algebraically continually from $-\pi$. We may suppose the point P to make any number of complete revolutions in the circle, then at every passage through any fixed position P_1, the number $x+iy$ has the same value, or an addition of a multiple of 2π to the argument leaves $x+iy$ unaltered. In other words, a variable

$$x+iy=r(\cos\theta+i\sin\theta),$$

considered as a function of its modulus r and its argument θ, is periodic with respect to the argument.

For any number $x+iy$, that value of θ which lies between the values $-\pi$ and π may be called the *principal value of the argument*; and we shall in general, in speaking of the argument of such a number, mean the principal value.

It should be observed that the principal value of the argument θ is not necessarily the principal value of $\tan^{-1}\frac{y}{x}$, as defined in Art. 38; for a given number $x+iy$, both $\cos\theta$ and $\sin\theta$ have given values, therefore θ has only one value between $-\pi$ and π

In this sense, the argument of a positive real number is 0, that of a positive imaginary number is $\frac{1}{2}\pi$, and of a negative imaginary number $-\frac{1}{2}\pi$. The principal value of the argument of a negative real number is, as defined above, ambiguous, being either π or $-\pi$; we shall however consider it to be π. The conjugate numbers $x+iy$, $x-iy$ have the same modulus, but their arguments are θ and $-\theta$. The modulus of $x+iy$ is frequently denoted by mod. $(x+iy)$, or also by $|x+iy|$.

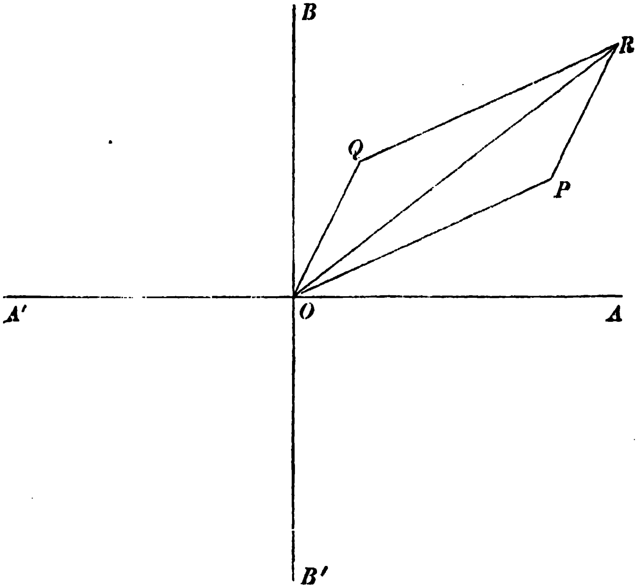
174. It is of fundamental importance to observe that whilst a real variable x can, whilst increasing continuously from x_1 to x_2, only pass through one set of values, this is not the case with a complex variable $x+iy$. There are an infinite number of ways in which such a variable may change continuously from $x_1 + iy_1$ to $x_2 + iy_2$, even supposing that both x and y continually increase, for the continuous increase of x from x_1 to x_2 is entirely independent of the increase of y from y_1 to y_2. This is essentially involved in the fact that two distinct unities are implied in a complex number, and is represented geometrically by the fact that two points P_1 and P_2 in the diagram may be joined in an infinite number of ways, the representative point moving along any arbitrary curve joining P_1 and P_2. If a real variable is to increase from x_1 to x_2, always remaining real, the representative point is restricted to remain in the x axis; if the variable is not restricted to have its intermediate values real, the representative point may describe any arbitrary curve drawn joining the two points on the x axis.

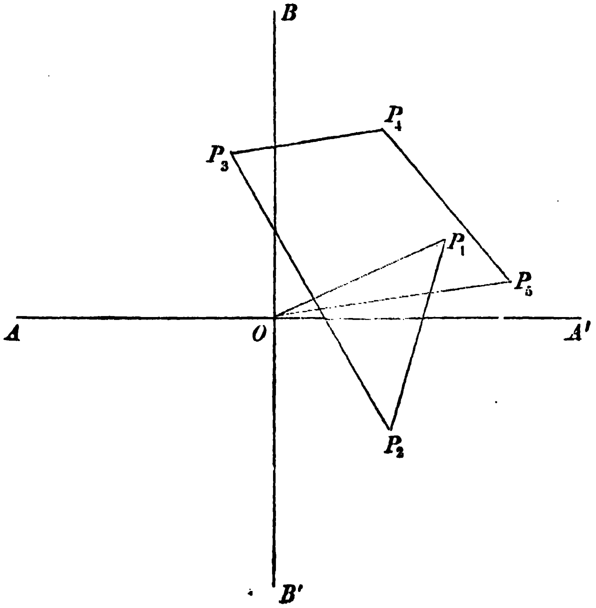
We may express this point by saying that a purely real or a purely imaginary number is essentially one-dimensional, whereas a complex number is two-dimensional, and requires a two-dimensional space for its geometrical representation.

The method of representing complex numbers geometrically was given by *Argand* in a tract published in 1806, but an earlier attempt at their representation had been made by Kühn in 1750. The theory founded on this method of representation was developed by Cauchy, Gauss, Riemann, and others, and forms the foundation of the modern theory of functions.

The addition of complex numbers.

175. Suppose two complex numbers $x_1 + iy_1$, $x_2 + iy_2$ are represented by the points P, Q; complete the parallelogram $OPRQ$, then the projection of OR on either axis is the sum of the projections of OP, PR, or of OP, OQ, on that axis; hence the point R represents the sum $(x_1 + x_2) + i(y_1 + y_2)$ of the two given complex numbers. We see therefore that the sum of two complex numbers is obtained geometrically by adding the straight lines, which represent those numbers, according to the parallelogram law. We have supposed that equal and parallel straight lines of the same length, and in the same direction, represent the same number, thus PR drawn from P parallel and equal to OQ represents $x_2 + iy_2$. We may therefore express the rule of addition thus: draw from O the straight line OP to represent $x_1 + iy_1$, and then from P draw PR to represent $x_2 + iy_2$, join OR, then OR, or the point R, represents the sum $(x_1 + x_2) + i(y_1 + y_2)$.


B
R
Q
P
A'
O
A
B'

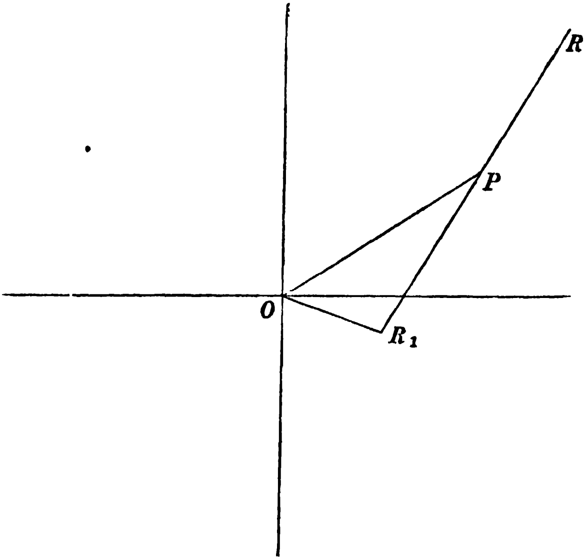

B
P_4
P_3
P_1
P_5
A
O
A'
P_2
B'

176. The mode of extension of the rule for addition, to any set of numbers, is now obvious.

Draw OP_1 in the second figure on page 228 to represent x_1+iy_1, then from P_1 draw P_1P_2 to represent x_2+iy_2, from P_2 draw P_2P_3 to represent x_3+iy_3, and so on; then join OP_n; the sum of the n numbers x_1+iy_1, x_2+iy_2, ... x_n+iy_n is represented by the straight line OP_n, or by the point P_n.

Since the length OP_n cannot be greater than the sum of the lengths OP_1, P_1P_2, ... $P_{n-1}P_n$, it follows that the modulus of the sum of a set of complex numbers is less than, or equal to, the sum of their moduli.

177. In order to subtract x_2+iy_2 from x_1+iy_1, a line PR_1 must be drawn from P to represent $-(x_2+iy_2)$, this will be equal to PR, and in the opposite direction; then the difference is represented by OR_1, or by the point R_1.



The multiplication of complex numbers.

178. The product of the two numbers

$$x_1+iy_1,\quad x_2+iy_2 \text{ is } (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1),$$

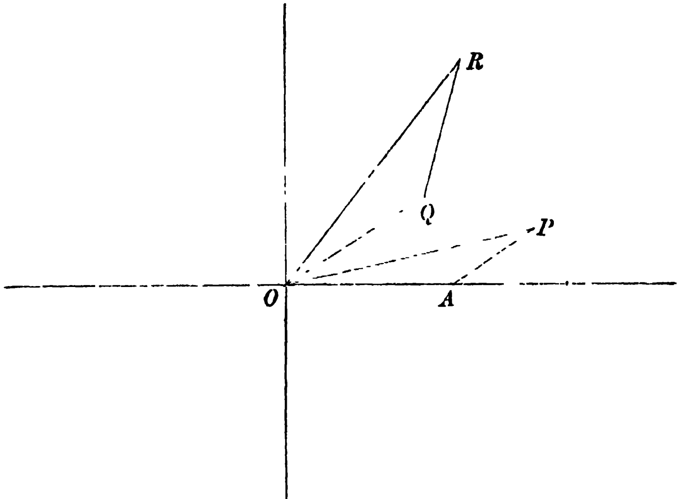
and if we replace the expressions by

$$r_1(\cos\theta_1+i\sin\theta_1),\quad r_2(\cos\theta_2+i\sin\theta_2),$$

their product may be written $r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$; this expression shews that the modulus of a product is equal to the product of the moduli, and the argument of the product is equal to the sum of the arguments of the two numbers.

It should however be observed that if θ_1, θ_2 are the principal values of the arguments of x_1+iy_1, x_2+iy_2, then $\theta_1+\theta_2$ is not necessarily the principal value of the argument of the product.

We can now obtain a geometrical construction for the product of two numbers; let A, P, Q represent the three numbers $+1$, x_1+iy_1, x_2+iy_2; join AP, on OQ describe a triangle QOR similar to AOP, and so that the angle QOR is equal to $+\theta_1$, then $ROA=\theta_1+\theta_2$, and also $OR:OQ::OP:OA$; hence the length of OR is equal to the product of the lengths of OP and OQ; it follows that the point R represents the product $(x_1+iy_1)(x_2+iy_2)$.



If we now introduce a third factor $x_3+iy_3=r_3(\cos\theta_3+i\sin\theta_3)$, we have

$$\begin{aligned}(x_1+iy_1)&(x_2+iy_2)(x_3+iy_3)\\&=r_1r_2r_3\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}\{\cos\theta_3+i\sin\theta_3\}\\&=r_1r_2r_3\{\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3)\},\end{aligned}$$

and we obtain, in a similar manner, the product of four or more complex numbers. In the case of n such numbers, we obtain the formula

$$(x_1 + iy_1)(x_2 + iy_2)\dots(x_n + iy_n)$$
$$= r_1 r_2 \dots r_n \{\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n)\} \dots(1).$$

Or *the modulus of the product of any set of complex numbers is the product of their moduli, and the argument of their product is the sum of their arguments.* The product may be obtained geometrically by a repeated application of the construction we have given for the product of two numbers.

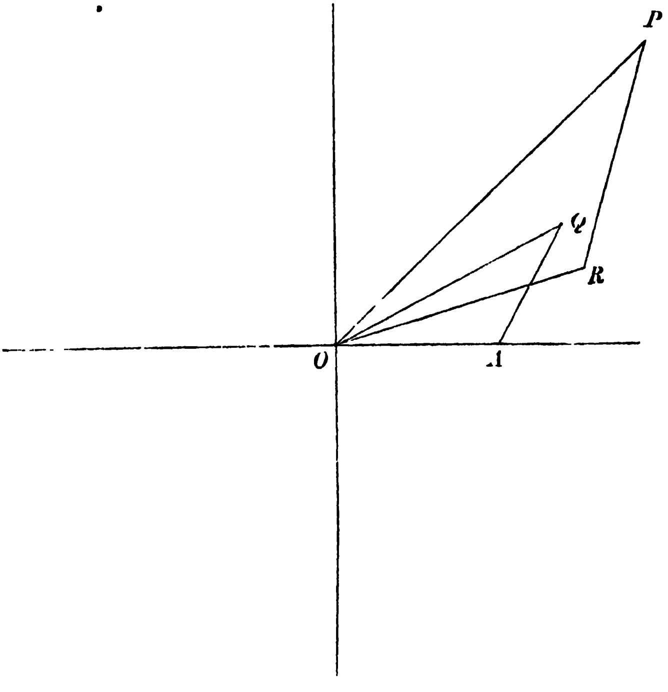
Division of one complex number by another.

179. The quotient $(x_1 + iy_1)(x_2 + iy_2)$ is equal to

$$\frac{1}{r_2^2}\{x_1x_2 + y_1y_2 - i(x_1y_2 - x_2y_1)\} \quad \text{or} \quad \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\};$$

thus the modulus of the quotient is the quotient of the moduli, and the argument of the quotient is the difference of the arguments of the two numbers.

To construct the quotient geometrically, join the point Q



(x_2+iy_2) to the point A $(+1)$, and draw a triangle ORP similar to the triangle OAQ, the angle ROP being measured equal to $-\theta_2$; then the angle ROA is $\theta_1-\theta_2$, and $OR=OP/OQ$, therefore the point R represents the quotient.

The powers of complex numbers.

180. If in equation (1), we put all the factors on the left-hand side of the equation equal to $x+iy$, we obtain the formula

$$(x+iy)^n=r^n(\cos n\theta+i\sin n\theta);$$

thus the modulus of the nth power of a complex number is the nth power of the modulus, and the argument is n times that of the given number. The number n here denotes any positive integer.

To construct such a power geometrically, let $P_1(x+iy)$ be joined to $A(+1)$; on OP_1 draw the triangle OP_1P_2 similar to OAP_1, on OP_2 draw OP_2P_3 similar to the same triangle, and so on; then the lengths of $OP_1, OP_2, \dots OP_n$ are $r, r^2, \dots r^n$, respectively, and the angles $P_1OA, P_2OA, \dots P_nOA$ are $\theta, 2\theta, \dots n\theta$, respectively, therefore the points $P_1, P_2, \dots P_n$ represent the numbers $(x+iy)$, $(x+iy)^2, \dots (x+iy)^n$.

In the particular case $r=1$, we have

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

and if Q_1 represents $\cos\theta+i\sin\theta$, then the points $Q_1, Q_2, \dots Q_n$, which represent the different powers of $\cos\theta+i\sin\theta$, are all on the circle of radius unity, and so that the arc between any two consecutive points of the series subtends an angle θ at the centre O.

181. In accordance with the theory of indices, supposing n to be a positive integer, the expression $(x+iy)^{\frac{1}{n}}$ denotes a number of which the nth power is $x+iy$. Now since the nth power of the modulus of a number is the modulus of its nth power, and since the modulus of any number is real and positive, the modulus of $(x+iy)^{\frac{1}{n}}$ is $\sqrt[n]{r}$, where $\sqrt[n]{r}$ is the real positive nth root of r. Suppose that $\sqrt[n]{r}(\cos\phi+i\sin\phi)$ is a value of $(x+iy)^{\frac{1}{n}}$, then we have

$$r(\cos\phi+i\sin\phi)^n=r(\cos\theta+i\sin\theta),$$

or $\cos n\phi + i \sin n\phi = \cos\theta + i\sin\theta$; therefore $\cos n\phi = \cos\theta$, and $\sin n\phi = \sin\theta$, or $n\phi = \theta + 2s\pi$, where s is any positive or negative integer including zero; hence a value of

$$(x+iy)^{\frac{1}{n}}$$

is
$$\sqrt[n]{r}\left\{\cos\frac{\theta+2s\pi}{n} + i\sin\frac{\theta+2s\pi}{n}\right\},$$

since the nth power of this expression is equal to $x+iy$. The above reasoning shews that every value of $(x+iy)^{\frac{1}{n}}$ must be of this form.

If we give s the values $0, 1, 2, \ldots n-1$, the expression

$$\cos\frac{\theta+2s\pi}{n} + i\sin\frac{\theta+2s\pi}{n}$$

has a different value for each of these values of s, for in order that it may have equal values for two values s_1, s_2 of s, we must have

$$\cos\frac{\theta+2s_1\pi}{n} = \cos\frac{\theta+2s_2\pi}{n}, \text{ and } \sin\frac{\theta+2s_1\pi}{n} = \sin\frac{\theta+2s_2\pi}{n},$$

whence $$\frac{\theta+2s_1\pi}{n} = 2k\pi = \frac{\theta+2s_2\pi}{n}, \text{ or } s_1 - s_2 = nk,$$

where k is some positive or negative integer; this cannot be the case if s_1 and s_2 are both less than n, and unequal, therefore the values are all different.

If we give s other values not lying between 0 and $n-1$, we shall obtain no more values of $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$, for if s_2 be such a value of s, it is always possible to find a number s_1 lying between 0 and $n-1$, such that $s_1 - s_2$ is a multiple of n, and therefore the value of the expression for $s = s_1$ is the same as for $s = s_2$.

We see then that all the values of $(x+iy)^{\frac{1}{n}}$ are given by the series of n numbers

$$\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),\quad \sqrt[n]{r}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right), \ldots\ldots$$
$$\sqrt[n]{r}\left\{\cos\frac{\theta+2(n-1)\pi}{n} + i\sin\frac{\theta+2(n-1)\pi}{n}\right\},$$

where $\sqrt[n]{r}$ is real and positive.

182. If θ be the principal value of the argument of $x+iy$, that is, that value of the argument which lies between $-\pi$ and π, we may regard $\sqrt[n]{r}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)$ as the *principal value* of $(x+iy)^{\frac{1}{n}}$. We may consider

$$\cos\frac{\theta}{n}+i\sin\frac{\theta}{n},\quad \cos\frac{\theta+2\pi}{n}+i\sin\frac{\theta+2\pi}{n},\quad \cos\frac{\theta+4\pi}{n}+i\sin\frac{\theta+4\pi}{n}$$

as the principal values of the nth roots of

$$\cos\theta+i\sin\theta,\ \cos(\theta+2\pi)+i\sin(\theta+2\pi),\ \cos(\theta+4\pi)+i\sin(\theta+4\pi)$$

respectively. The different values of $(x+iy)^{\frac{1}{n}}$ are then the principal values of the corresponding expression in r and θ when n different values of the argument θ are taken, the principal value of $(x+iy)^{\frac{1}{n}}$ being considered as that expression in which θ has its principal value.

The two values of $a^{\frac{1}{2}}$, where a is a positive real quantity, are $\sqrt{a}(\cos 0+i\sin 0)$ and $\sqrt{a}(\cos\pi+i\sin\pi)$, that is $\sqrt{a}$ and $-\sqrt{a}$, where $\sqrt{a}$ is the positive square root of a. The values of $(-a)^{\frac{1}{2}}$, in which case $\theta=\pi$, are $\sqrt{a}(\cos\frac{1}{2}\pi+i\sin\frac{1}{2}\pi)$, $\sqrt{a}(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi)$, or $i\sqrt{a}$, $-i\sqrt{a}$. The principal value of $a^{\frac{1}{2}}$ is $\sqrt{a}$, and of $(-a)^{\frac{1}{2}}$ is $i\sqrt{a}$.

183. The nth roots of unity are obtained from the expressions in Art. 181 by putting $r=1$, $\theta=0$; they are therefore

$$1,\quad \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\quad \cos\frac{4\pi}{n}+i\sin\frac{4\pi}{n},$$
$$\ldots\ldots\ldots\ \cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)\pi}{n}.$$

If we denote by ω the root $\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, the whole of the roots are given by the series $1, \omega, \omega^2, \ldots \omega^{n-1}$.

Since

$$\cos\frac{\theta+2r\pi}{n}+i\sin\frac{\theta+2r\pi}{n}=\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)\left(\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}\right),$$

it follows that, if $\sqrt[n]{x+iy}$ denote the principal value of $(x+iy)^{\frac{1}{n}}$, then all the values are given by the series

$$\sqrt[n]{x+iy},\ \omega\sqrt[n]{x+iy},\ \omega^2\sqrt[n]{x+iy},\ldots\ldots\ \omega^{n-1}\sqrt{x+iy}.$$

EXAMPLES.

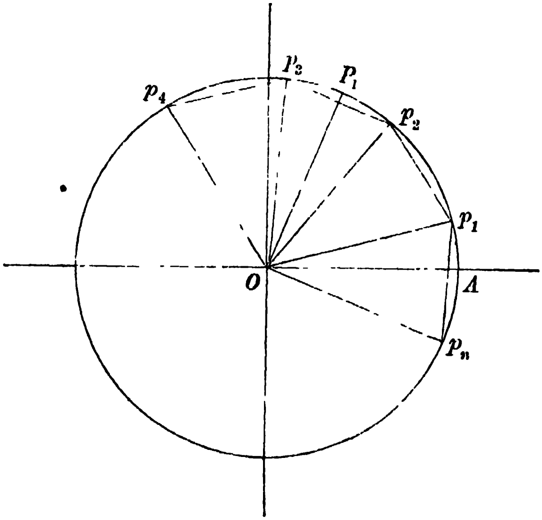
(1) *Find all the values of* $(-1)^{\frac{1}{5}}$ *and of* $(-1)^{\frac{1}{8}}$.

(2) *Find the values of* $(1+\sqrt{-1})^{\frac{1}{5}}$.

184. We shall now shew how to represent geometrically the nth roots of a complex number; the method will give an intuitive proof of the existence of n different values of the nth root. Without any loss of generality we may take the modulus to be unity, so that we have to represent the values of

$$(\cos\theta + i\sin\theta)^{\frac{1}{n}}.$$

Let a point P describe the circle of radius unity starting from A, at which $\theta = 0$, then in any position of P for which the angle POA described by OP is θ, the point P represents the expression $\cos\theta + i\sin\theta$. Let another point p start from A at the same time as P, and let its angular velocity be always equal to $1/n$ of that of P, so that the angle pOA is always equal



to θ/n, then p represents $\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$. When P reaches any position P_1 for the first time, let p be at p_1, then the angle P_1OA is n times the angle p_1OA, therefore P_1 represents the nth power of the number represented by p_1, or conversely p_1 represents an nth root of $\cos\theta_1 + i\sin\theta_1$. Now let P move round the circle until it again reaches P_1, so that it has described the angle $\theta_1 + 2\pi$, then p will be at p_2, where p_2OA is equal to $(\theta_1 + 2\pi)/n$; if P proceeds to make another complete revolution, when it again reaches the position P_1, p will be at p_3, where $p_3OA = (\theta_1 + 4\pi)/n$, and so on. The points $p_1, p_2, \dots p_n$ are the angular points of a regular polygon of n sides inscribed in the circle. When P makes more than n complete revolutions round O, the point p will again reach the positions $p_1, p_2, \dots$. Each of the points $p_1, p_2, \dots p_n$ represents a value of $(\cos\theta_1 + i\sin\theta_1)^{\frac{1}{n}}$, since the nth power of the expressions represented by any one of these points is the expression represented by the point P. The point p_1 represents the value for the smallest argument θ_1. We have thus obtained the n values of $(\cos\theta_1 + i\sin\theta_1)^{\frac{1}{n}}$, and we see that these values are the different values of $\cos\frac{\theta_1 + 2s\pi}{n} + i\sin\frac{\theta_1 + 2s\pi}{n}$, when $s = 0, 1, 2, \dots n-1$.

185. To obtain graphically the nth roots of any number $x + iy$, we must be able (1) to divide an angle into n equal parts, and (2) to inscribe a regular polygon of n sides in a circle, and (3) in order to construct the modulus, we must be able to construct a straight line whose length is the nth root of the length of a given line. In order to obtain all the nth roots of unity, it is only necessary to solve the second of these geometrical problems, since in this case the angle to be divided into n parts is zero. The problem of inscribing a regular polygon of n sides in a given circle is therefore equivalent to that of obtaining the numerical values of the roots of the equation $x^n - 1 = 0$. This geometrical problem can be solved by a method involving the construction only of straight lines and circles in the following cases:

(1) When n is a power of 2; for example $n = 4, 8, 16, 32$.

(2) When n is a prime number of the form $2^m + 1$; for example, when $n = 3, 5, 17, 257$. This was proved by Gauss in his *Disquisitiones arithmeticae.*

(3) When n is the product of different prime numbers of the form $2^m + 1$, and of any power of 2; for example, when $n = 15$, 85, 255.

The proof of Gauss' theorem would lead us too far into the theory of numbers; we have however considered the special case $n = 17$ in Art. 85, Ex. (4), where $\sin\pi/17$ is found in a form involving radicals.

De Moivre's theorem.

186. *For all real values of* m, *$\cos m\theta + i \sin m\theta$ is a value of $(\cos\theta + i\sin\theta)^m$.*

This theorem, known as De Moivre's theorem, has been proved in Arts. 180 and 181, in the two cases $m = n$, and $m = 1/n$, where n is a positive integer. To complete the proof, we have to consider the cases when $m = p/q$, a positive fraction, when m is a positive irrational number, and lastly when m is any negative real number. It is clear that $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = (\cos p\theta + i\sin p\theta)^{\frac{1}{q}}$, and one value of this is $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$. Therefore the theorem holds when m is a positive rational number.

It should be remarked that all the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ are given by the expression

$$\cos\frac{p(\theta + 2s\pi)}{q} + i\sin\frac{p(\theta + 2s\pi)}{q},$$

where $s = 0, 1, 2, \ldots q-1$, when p/q is a rational fraction in its lowest terms.

When m is not a rational number, it can always be defined in an indefinite number of ways as the limit of a convergent sequence of rational numbers $m_1, m_2, \ldots m_s, \ldots$. Such a convergent sequence is characterized by the property that, if ϵ be an arbitrarily chosen rational number, as small as we please, s can always be so determined that m_s differs arithmetically from each of the subsequent numbers $m_{s+1}, m_{s+2}, \ldots$ by less than ϵ. If r is any positive real number, the principal value of r^m is defined as the limit of the convergent sequence $r^{m_1}, r^{m_2}, \ldots r^{m_s}, \ldots$, when each of the numbers is real and positive, r^{m_s} having its principal value. It is known[1] that this sequence is convergent, and that it has a limit which is independent of the particular sequence of rational numbers employed to define the irrational number m.

If z denotes the complex number $r(\cos\theta + i\sin\theta)$, a value of z^m, when m is an irrational number, is defined as the limit of the sequence of numbers $r^{m_1}(\cos\theta + i\sin\theta)^{m_1}, r^{m_2}(\cos\theta + i\sin\theta)^{m_2}, \ldots$

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[1] For a proof of this, see the author's *Theory of functions of a real variable*, p. 44. In Chapter I of that work, a full discussion of the theory of irrational numbers is given.

$r^{m_s}(\cos\theta + i\sin\theta)^{m_s}$, ..., where r^{m_s} has its principal value, and corresponding values for all values of s are assigned to $(\cos\theta + i\sin\theta)^{m_s}$. In accordance with this definition, one value of z^m is the limit of the sequence $r^{m_1}(\cos m_1\theta + i\sin m_1\theta)$, $r^{m_2}(\cos m_2\theta + i\sin m_2\theta)$, ... $r^{m_s}(\cos m_s\theta + i\sin m_s\theta)$, ... Since r^{m_s} converges to r^m, and $\cos m_s\theta + i\sin m_s\theta$ converges to $\cos m\theta + i\sin m\theta$, on account of the fact that $\cos m\theta$, $\sin m\theta$ are continuous functions of m, we see that one value of z^m is $r^m(\cos m\theta + i\sin m\theta)$; and one value of $(\cos\theta + i\sin\theta)^m$ is $\cos m\theta + i\sin m\theta$. Thus De Moivre's theorem is established for a positive irrational index.

The general values of $(\cos\theta + i\sin\theta)^m$ are

$$\cos m(\theta + 2s\pi) + i\sin m(\theta + 2s\pi),$$

where s denotes any positive or negative integer. Since $m(s_1 - s_2)$ can never be an integer when m is irrational, we see that $(\cos\theta + i\sin\theta)^m$ has an indefinitely great set of values.

It can be shewn that the definition of z^m, in accordance with which its values are those of $r^m\{\cos m(\theta + 2s\pi) + i\sin m(\theta + 2s\pi)\}$ is such that the laws of indices applicable to real indices still hold for irrational indices.

In case m has a negative rational or irrational value $-k$, we have $(\cos\theta + i\sin\theta)^m = 1/(\cos\theta + i\sin\theta)^k$; and one value of this is always $1/(\cos k\theta + i\sin k\theta)$, or $\cos k\theta - i\sin k\theta$, which is equal to $\cos m\theta + i\sin m\theta$. Thus De Moivre's theorem holds for any negative index.

187. The theorem

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\dots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n),$$

used in the proof of De Moivre's theorem, affords a proof of the theorems (28), (29), (30) of Art. 49. We may write the left-hand side of this identity in the form

$$\cos\theta_1\cos\theta_2\dots\cos\theta_n(1 + i\tan\theta_1)(1 + i\tan\theta_2)\dots(1 + i\tan\theta_n);$$

hence equating the real and imaginary parts on both sides of the identity, we have

$$\cos(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1\cos\theta_2\dots\cos\theta_n(1 - t_2 + t_4 - \dots),$$
$$\sin(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1\cos\theta_2\dots\cos\theta_n(t_1 - t_3 + t_5 - \dots),$$

where t_s denotes the sum of the products of the n tangents taken s at a time.

The theorems (39), (40), (43), of Art. 51, are obtained at once from the theorem $\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$, by expanding the right-hand side of the equation by the Binomial theorem, and equating the real and imaginary parts on both sides of the equation.

When n is a positive integer, we have $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, and therefore also $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$; thence we obtain the formulae

$$\cos n\theta = \tfrac{1}{2}(\cos\theta + i\sin\theta)^n + \tfrac{1}{2}(\cos\theta - i\sin\theta)^n,$$
$$i\sin n\theta = \tfrac{1}{2}(\cos\theta + i\sin\theta)^n - \tfrac{1}{2}(\cos\theta - i\sin\theta)^n.$$

The first of these equations is really an expression of the fact mentioned in Art. 51, that $1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^n\cos n\theta + \ldots$ is a recurring series of which $1 - 2x\cos\theta + x^2$ is the scale of relation. Denoting $\cos n\theta$ by u_n, we have $u_n - 2\cos\theta \,.\, u_{n-1} + u_{n-2} = 0$; to solve this equation assume, as usual in such cases, $u_n = Ak^n$, then we obtain for k the quadratic $k^2 - 2k\cos\theta + 1 = 0$, of which the roots are $k = \cos\theta \pm i\sin\theta$, hence

$$u_n = A(\cos\theta + i\sin\theta)^n + B(\cos\theta - i\sin\theta)^n$$

is the complete solution of the equation for u_n. Putting $n = 1$, and $n = 2$, we find $A = B = \frac{1}{2}$, and thus obtain the expression given above for $\cos n\theta$. The expression for $\sin n\theta$ may be found in a similar manner.

Factorization.

188. We are now in a position to resolve $x^n - (a + ib)$ into n factors linear with respect to x. The expression vanishes if x is equal to any one of the values of $(a + ib)^{\frac{1}{n}}$; if $q_1, q_2, \ldots q_n$ denote the n values of this expression, we shall have

$$x^n - (a + ib) = (x - q_1)(x - q_2)\ldots(x - q_n),$$

for since $x^n - (a + ib)$ vanishes when $x - q_s = 0$, $x - q_s$ must be a factor without remainder; thus we obtain n different factors and there can obviously be no more. Put $a = r\cos\theta, b = r\sin\theta$, then the expression for $x^n - (a + ib)$ in factors becomes

$$\prod_{s=0}^{s=n-1}\left\{x - \rho\left(\cos\frac{\theta + 2s\pi}{n} + i\sin\frac{\theta + 2s\pi}{n}\right)\right\},$$

where $\rho = \sqrt[n]{r} = (a^2 + b^2)^{\frac{1}{2n}}$.

From this result several of the factorizations already obtained in Chap. VII may be deduced.

(1) Let $a=1$, $b=0$, we then obtain

$$x^n-1=\prod_{s=0}^{s=n-1}\left(x-\cos\frac{2s\pi}{n}-i\sin\frac{2s\pi}{n}\right),$$

and since
$$\frac{2s\pi}{n}+\frac{2(n-s)\pi}{n}=2\pi,$$

this gives us, if n is odd,

$$x^n-1=(x-1)\prod_{s=1}^{s=\frac{1}{2}(n-1)}\left(x-\cos\frac{2s\pi}{n}-i\sin\frac{2s\pi}{n}\right)\left(x-\cos\frac{2s\pi}{n}+i\sin\frac{2s\pi}{n}\right)$$

$$=(x-1)\prod_{s=1}^{s=\frac{1}{2}(n-1)}\left(x^2-2x\cos\frac{2s\pi}{n}+1\right)$$

and
$$x^n-1=(x-1)(x+1)\prod_{s=1}^{s=\frac{1}{2}(n-2)}\left(x^2-2x\cos\frac{2s\pi}{n}+1\right),$$
if n is even.

(2) Let $a=-1$, $b=0$, then we obtain the formulae

$$x^n+1=(x+1)\prod_{s=0}^{s=\frac{1}{2}(n-3)}\left(x^2-2x\cos\frac{(2s+1)\pi}{n}+1\right),\quad (n\text{ odd}),$$

$$x^n+1=\prod_{s=0}^{s=\frac{1}{2}(n-2)}\left(x^2-2x\cos\frac{(2s+1)\pi}{n}+1\right),\quad (n\text{ even}).$$

(3) $x^{2n}-2x^n\cos\theta+1$

$$=(x^n-\cos\theta-i\sin\theta)(x^n-\cos\theta+i\sin\theta)$$

$$=\prod_{s=0}^{s=n-1}\left(x-\cos\frac{\theta+2s\pi}{n}-i\sin\frac{\theta+2s\pi}{n}\right)\left(x-\cos\frac{\theta+2s\pi}{n}+i\sin\frac{\theta+2s\pi}{n}\right)$$

$$=\prod_{s=0}^{s=n-1}\left(x^2-2x\cos\frac{\theta+2s\pi}{n}+1\right),$$

or writing x/y for x, and multiplying both sides by y^{2n}, we have

$$x^{2n}-2x^ny^n\cos\theta+y^{2n}=\prod_{s=0}^{s=n-1}\left(x^2-2xy\cos\frac{\theta+2s\pi}{n}+y^2\right).$$

(4) From the last result we have

$$x^n+x^{-n}-2\cos\theta=\prod_{s=0}^{s=n-1}\left(x+x^{-1}-2\cos\frac{\theta+2s\pi}{n}\right).$$

Put $x=\cos\phi+i\sin\phi$, then $x^{-1}=\cos\phi-i\sin\phi$,

and $x^n=\cos n\phi+i\sin n\phi$, $x^{-n}=\cos n\phi-i\sin n\phi$

therefore, changing θ into $n\theta$,

$$\cos n\phi-\cos n\theta=2^{n-1}\prod_{s=0}^{s=n-1}\left\{\cos\phi-\cos\left(\theta+\frac{2s\pi}{n}\right)\right\}.$$

Properties of the circle.

189. Certain well-known properties of the circle may be obtained by means of the factorization formulae of the last Article. Let $A_1A_2A_3 \ldots A_n$ be a regular polygon of n sides inscribed in a circle of radius a, and let P be any point in the plane of the circle, its distance from O, the centre of the circle, being denoted by c. Let the angle POA_1 be denoted by θ, then the angles POA_2, POA_3, ... are $\theta + 2\pi/n$, $\theta + 4\pi/n$, ... respectively. Then

$$PA_1^2 . PA_2^2 . PA_3^2 \ldots PA_n^2 = \prod_{s=0}^{s=n-1} \left\{a^2 - 2ac \cos\left(\theta + \frac{2r\pi}{n}\right) + c^2\right\},$$

hence we have the theorem

$$PA_1^2 . PA_2^2 . PA_3^2 \ldots PA_n^2 = a^{2n} - 2a^n c^n \cos n\theta + c^{2n},$$

which is known as *De Moivre's* property of the circle.

In the case when P is on the circumference, the theorem becomes

$$PA_1 . PA_2 \; PA_3 \ldots PA_n = 2a^n \sin \tfrac{1}{2} n\theta.$$

In the case when P is on the radius OA_1, we have $\theta = 0$, and the theorem becomes

$$PA_1 . PA_2 \ldots PA_n = a^n \sim c^n.$$

Again if P lies on the bisector of the angle A_nOA_1, we have $\theta = \pi/n$, and the theorem becomes

$$PA_1 . PA_2 \ldots PA_n = a^n + c^n.$$

The last two cases are known as *Cotes'* properties of the circle.

190. EXAMPLES.

(1) *Express* $x^{m-1}/(1+x^n)$ *in partial fractions,* m *being an integer less than* n.

If a be a root of the equation $x^n + 1 = 0$, the partial fraction corresponding to the factor $x - a$ is $\dfrac{a^{m-1}}{na^{n-1}} \cdot \dfrac{1}{x-a}$, or $\dfrac{1}{n}\dfrac{a^{m-n}}{x-a}$; taking the two fractions corresponding to the conjugate values of a, $\cos\dfrac{2r+1}{n}\pi \pm i \sin\dfrac{2r+1}{n}\pi$, together, we obtain the fraction

$$\frac{1}{n}\frac{2x\cos\dfrac{2r+1}{n}(n-m)\pi - 2\cos\dfrac{2r+1}{n}(n-m+1)\pi}{x^2 - 2x\cos\dfrac{2r+1}{n}\pi + 1}$$

or

$$\frac{2}{n} \cdot \frac{\cos(2r+1)\dfrac{m-1}{n}\pi - x\cos(2r+1)\dfrac{m}{n}\pi}{x^2 - 2x\cos\dfrac{2r+1}{n}\pi + 1};$$

if n is odd, we have the additional fraction $\dfrac{(-1)^{n-m}}{n(x+1)}$; hence when n is odd

$$\frac{x^{m-1}}{1+x^n}=\frac{(-1)^{n-m}}{n(x+1)}+\frac{2}{n}\sum_{r=0}^{r=\frac{1}{2}(n-3)}\frac{\cos(2r+1)\dfrac{m-1}{n}\pi-x\cos(2r+1)\dfrac{m}{n}\pi}{x^2-2x\cos\dfrac{2r+1}{n}\pi+1}$$

and when n is even

$$\frac{x^{m-1}}{1+x^n}=\frac{2}{n}\sum_{r=0}^{r=\frac{1}{2}n-1}\frac{\cos(2r+1)\dfrac{m-1}{n}\pi-x\cos(2r+1)\dfrac{m}{n}\pi}{x^2-2x\cos\dfrac{2r+1}{n}\pi+1}$$

(2) *Express* $x^{m-1}/(x^n-1)$ *in partial fractions,* m *being less than* n.

(3) *Prove that*

$$\frac{x^n-a^n\cos n\theta}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}=\frac{1}{nx^{n-1}}\sum_{r=0}^{r=n-1}\frac{x-a\cos\left(\theta+\dfrac{2r\pi}{n}\right)}{x^2-2xa\cos\left(\theta+\dfrac{2r\pi}{n}\right)+a^2}.$$

The denominator of the fraction $\dfrac{n(x^{2n-1}-a^nx^{n-1}\cos n\theta)}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}$ is resolved into factors, and the fraction corresponding to each factor can then be determined as in Ex. (1).

(4) *Prove that*

(*a*) $$\frac{n\sin n\theta}{\sin\theta}\cdot\frac{1}{\cos n\theta-\cos n\phi}=\sum_{r=0}^{r=n-1}\frac{1}{\cos\theta-\cos(\phi+2r\pi/n)};$$

(*b*) $$\frac{n^2\sin n\theta\sin n\phi}{\sin\theta}\cdot\frac{1}{(\cos n\theta-\cos n\phi)^2}=\sum_{r=0}^{r=n-1}\frac{\sin(\phi+2r\pi/n)}{\{\cos\theta-\cos(\phi+2r\pi/n)\}^2}.$$

The expression on the left-hand side in (*a*) is an algebraical function of $\cos\theta$, and can therefore be resolved into partial fractions, as in Ex. (1); the equation (*b*) is obtained by differentiating both sides of (*a*) with respect to ϕ, or what amounts to the same thing, by changing ϕ into $\phi+h$ and equating the coefficients of h, on both sides of the equation.

(5) *Shew that if*

$$\cos\theta+\cos\phi+\cos\psi=0,\quad \textit{and}\quad \sin\theta+\sin\phi+\sin\psi=0,$$

then $$\cos 3\theta+\cos 3\phi+\cos 3\psi-3\cos(\theta+\phi+\psi)=0,$$

and $$\sin 3\theta+\sin 3\phi+\sin 3\psi-3\sin(\theta+\phi+\psi)=0.$$

This is an example of the general method of deducing trigonometrical theorems from algebraical ones, by substituting complex values for the letters. If $a+b+c=0$, we have $a^3+b^3+c^3-3abc=0$; let $a=\cos\theta+i\sin\theta$, $b=\cos\phi+i\sin\phi$, $c=\cos\psi+i\sin\psi$, then we have given that if

$$(\cos\theta+\cos\phi+\cos\psi)+i(\sin\theta+\sin\phi+\sin\psi)=0,$$

$$(\cos 3\theta+\cos 3\phi+\cos 3\psi)+i(\sin 3\theta+\sin 3\phi+\sin 3\psi)$$
$$-3\{\cos(\theta+\phi+\psi)+i\sin(\theta+\phi+\psi)\}=0;$$

equating to zero the real and imaginary parts separately in each equation, the theorem follows.

EXAMPLES ON CHAPTER XIII.

1. Prove that $\left(\dfrac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\cos\left(\tfrac{1}{2}n\pi-n\phi\right)+i\sin\left(\tfrac{1}{2}n\pi-n\phi\right)$.

2. Evaluate

$$\{\cos\theta-\cos\phi+i(\sin\theta-\sin\phi)\}^n+\{\cos\theta-\cos\phi-i(\sin\theta-\sin\phi)\}^n.$$

3. Prove that

$$\frac{(1+x)^n-(1-x)^n}{2x}=A\left(x^2+\tan^2\frac{\pi}{n}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\ldots\ldots\left(x^2+\tan^2\frac{r\pi}{n}\right),$$

where $r=\frac{1}{2}(n-1)$ or $\frac{1}{2}n-1$, and A is 1 or n, according as n is odd or even.

4. Prove that

$$4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\,\Sigma\sin(p\alpha+q\beta+r\gamma)$$
$$=\sin\{(n+1)\alpha-\tfrac{1}{2}(\beta+\gamma)\}\sin\tfrac{1}{2}(\beta-\gamma)+\ldots,$$

where Σ denotes the sum taken for all positive integral values of p, q, r (including zero), such that $p+q+r=n$.

5. If p is a positive integer and $\alpha, \beta, \gamma \ldots$ are the roots of the equation $x^p=1$, and n is any numerical quantity greater than unity, shew that the only real value of $\alpha^{\frac{1}{n}}+\beta^{\frac{1}{n}}+\gamma^{\frac{1}{n}}+\ldots$ is $\tan\dfrac{\pi}{n}\Big/\tan\dfrac{\pi}{pn}$.

6. If $\qquad (1+x)^n=p_0+p_1x+p_2x^2+\ldots\ldots,$

prove that

$$p_0-p_2+p_4-\ldots\ldots=2^{\frac{1}{2}n}\cos\tfrac{1}{4}n\pi,$$
$$p_1-p_3+p_5-\ldots\ldots=2^{\frac{1}{2}n}\sin\tfrac{1}{4}n\pi.$$

7. If $x_1, x_2, \ldots x_n$ be the corresponding roots selected from the conjugate pairs of roots of the equation $x^{2n}-2x^n\cos n\theta+1=0$, and if

$$f(\alpha)=\sum_{r=1}^{r=n}x_r\cos\left(\alpha+\frac{r\pi}{n}\right),$$

prove that

$$f(\alpha_1)f(\alpha_2)\ldots\ldots f(\alpha_p)=\left(\tfrac{1}{2}n\right)^{p-1}\left[f\left\{\frac{1}{p}(\alpha_1+\alpha_2+\ldots+\alpha_p)\right\}\right]^p.$$

8. If $\alpha, \beta, \gamma, \delta, \epsilon$ be any five angles such that the sum of their cosines and also the sum of their sines is zero, shew that

$$\Sigma\cos 4\alpha=\tfrac{1}{2}(\Sigma\cos 2\alpha)^2-\tfrac{1}{2}(\Sigma\sin 2\alpha)^2,$$
$$\Sigma\sin 4\alpha=\Sigma\sin 2\alpha\,.\,\Sigma\cos 2\alpha.$$

9. If $t_1, t_2, \ldots t_n$ be the sum of the products of the n quantities $\tan x$, $\tan 2x$, $\tan 2^2x$, ... $\tan 2^{n-1}x$, taken 1, 2, 3, ... n together, prove that

$$1-t_2+t_4-t_6+\ldots=2^n\sin x\cos(2^n-1)x\operatorname{cosec}2^nx,$$
$$t_1-t_3+t_5-\ldots=2^n\sin x\sin(2^n-1)x\operatorname{cosec}2^nx.$$

10. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, shew that

$$\cos n\alpha+\cos n\beta+\cos n\gamma$$

is equal to zero unless n is a multiple of 3, and if n is a multiple of 3, it is equal to $3\cos\frac{1}{3}n(\alpha+\beta+\gamma)$.

11. Prove that the values of x which satisfy the equation

$$1-nx-\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+(-1)^{\frac{1}{2}n(n+1)}x^n=0$$

are $x=\tan\frac{(4r+1)\pi}{4n}$, where r is any integer.

12. Prove that

$$\sum_{r=1}^{r=n}(-1)^{r-1}\frac{\sin^2 ra\cos^{2n-3}ra}{x^2+\tan^2 ra}=\frac{(2n+1)x}{(1+x)^{2n+1}-(1-x)^{2n+1}},$$

where $$a=\frac{\pi}{2n+1}.$$

13. If ${}_sP_r$ denotes the sum of the products taken s together of the quantities

$$\tan^2\pi/(2n+1),\ \tan^2 2\pi/(2n+1),\ \ldots\ldots\ \tan^2 n\pi/(2n+1),$$

the quantity $\tan^2 r\pi/(2n+1)$ being omitted, and if

$$A_r=(-1)^{r-1}\sin^2 r\pi/(2n+1)\,.\,\cos^{2n-3}r\pi/(2n+1),$$

prove that $\Sigma A_r\,.\,{}_sP_r=0$, the summation extending to all values of r from 1 to n, and s having any value from 1 to n.

14. A regular polygon of n sides is inscribed in a circle, and from any point on the circumference chords are drawn to the angular points; if these chords are denoted by $c_1, c_2, \ldots c_n$ (beginning with the chord drawn to the nearest angular point and taking the rest in order), prove that the quantity $c_1c_2+c_2c_3+\ldots+c_{n-1}c_n+c_nc_1$ is independent of the position of the point from which the chords are drawn.

15. If $A_1A_2\ldots A_{2n+1}$ are the angular points of a regular polygon inscribed in a circle, and O is any point on the circumference between A_1 and A_{2n+1}, prove that the sum of the lengths $OA_1, OA_3, \ldots OA_{2n+1}$ is equal to the sum of $OA_2, OA_4, \ldots OA_{2n}$.

16. If $\rho_1, \rho_2, \ldots \rho_n$ are the distances of a point P in the plane of a regular polygon from the vertices, prove that

$$\sum_1^n\frac{1}{\rho^2}=\frac{n}{r^2-a^2}\cdot\frac{r^{2n}-a^{2n}}{r^{2n}-2r^na^n\cos n\theta+a^{2n}},$$

where a is the radius of the circle round the polygon, r is the distance of P from O, and θ the angle OP makes with the radius to any vertex of the polygon.

17. Straight lines whose lengths are successively proportional to $1, 2, 3\ldots n$, form a rectilineal figure whose exterior angles are each equal to $2\pi/n$; if a polygon be formed by joining the extremities of the first and last lines, shew that its area is

$$\frac{n(n+1)(2n+1)}{24}\cot\frac{\pi}{n}+\frac{16}{n}\cot\frac{\pi}{n}\operatorname{cosec}^2\frac{\pi}{n}.$$

18. The regular polygon $A_1A_2A_3 \ldots A_{2m}$ has $2m$ sides; shew that the product of the perpendiculars from the centre of the circumscribed circle on A_1A_2, A_1A_3, ... A_1A_m is $(\frac{1}{2}a)^{m-1}\sqrt{m}$.

19. Shew that if $A_1A_2 \ldots A_{2n}$, $B_1B_2 \ldots B_{2n}$ be two concentric and similarly situated regular polygons of $2n$ sides, then

$$\frac{PA_1 . PA_3 . \ldots PA_{2n-1}}{PA_2 . PA_4 . \ldots PA_{2n}} = \frac{PB_1 . PB_3 . \ldots PB_{2n-1}}{PB_2 . PB_4 . \ldots PB_{2n}},$$

where P is anywhere on the concentric circle whose radius is a mean proportional between the radii of the circles circumscribing the polygons.

20. A point O is taken within a circle of radius a, at a distance b from the centre, and points P_1, P_2, ... P_n are taken on the circumference so that P_1P_2, P_2P_3, ... P_nP_1 subtend equal angles at O; prove that

$$OP_1 + OP_2 + \ldots + OP_n = (a^2 - b^2)(OP_1^{-1} + OP_2^{-1} + \ldots + OP_n^{-1}).$$

21. Prove that if n is a positive integer

$$\cos n\theta = 1 + 2n \sin\frac{\theta}{2} \cos\frac{\theta+\pi}{2} + \frac{n(n-1)}{2!} 2^2 \sin^2\frac{\theta}{2} \cos\frac{2(\theta+\pi)}{2}$$
$$+ \frac{n(n-1)(n-2)}{3!} 2^3 \sin^3\frac{\theta}{2} \cos\frac{3(\theta+\pi)}{2} + \ldots.$$

22. Shew that the number m of distinct regular polygons of n sides which can be inscribed in a given circle of radius r is equal to half the number of integers less than n and prime to it.

Shew also that the product of their sides is equal to $r^m\sqrt{n}/\sqrt{n-2m}$, or r^m, according as n is, or is not, the power of a prime number.

CHAPTER XIV.

THE THEORY OF INFINITE SERIES.

191. We shall, in this Chapter, give some propositions concerning the convergence of infinite series in which the terms are real or complex numbers, or variables. Anything like a complete account of the theory of such series would be beyond the limits of this work; we shall therefore confine ourselves to what is absolutely necessary for the purpose of discussing the nature and properties of trigonometrical series.

The convergence of real series.

192. Let $a_1, a_2, a_3, \ldots a_n, \ldots$ be a sequence of real numbers formed according to any prescribed law, and let

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n.$$

If S_n has a definite finite limit S, when n is indefinitely increased, the infinite series $a_1 + a_2 + a_3 + \ldots$ is said to be convergent, and S is said to be its limiting sum, or simply its sum.

We shall, in this Chapter, use the notation LS_n to denote the limit of S_n when n is indefinitely increased, whenever that limit exists.

The condition that $LS_n = S$ is that, corresponding to each arbitrarily chosen positive number ϵ, as small as we please, a value n_ϵ of n can be determined such that the arithmetical value of $S - S_n$ is less than ϵ, for every value of n which is $\geqq n_\epsilon$.

When the series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ converges to S, the series $a_{n+1} + a_{n+2} + \ldots$ is convergent, and its limiting sum is $S - S_n$, which may be denoted by R_n. The number R_n is called the remainder of the convergent series $a_1 + a_2 + \ldots + a_n + \ldots$, after n terms, and the remainders $R_1, R_2, \ldots R_n, \ldots$ form a sequence of numbers such that $LR_n = 0$. It should be observed that it is only on the assumption of the convergence of the series that the remainders R_n have any meaning.

The number $a_{n+1}+a_{n+2}+\dots+a_{n+m}$ may be denoted by $R_{n,m}$; and the numbers $R_{n,1}$, $R_{n,2}$, $R_{n,3}$, ... are called the partial remainders of the series after n terms. It will be observed that these partial remainders $R_{n,m}$ exist as definite numbers for all values of n and m, whether the given series is convergent or not.

The limiting sum of a convergent series $a_1+a_2+\dots a_n+\dots$ is frequently denoted by $\sum\limits_1^\infty a$.

193. A series $a_1+a_2+a_3+\dots+a_n+\dots$ may be such that the numbers S_n have no definite limit as n is increased indefinitely. The following cases may arise:

(1) It may happen that, corresponding to each arbitrarily chosen positive number k, as great as we please, a value n_k of n can be determined such that all the numbers S_{n_k}, S_{n_k+1}, ... S_{n_k+m}, ... are of the same sign, and are all numerically greater than k. In this case S_n increases indefinitely with n, either in the positive or in the negative direction; the series is then said to be divergent. The fact of the divergence is then sometimes denoted by $LS_n=\infty$, or $LS_n=-\infty$. as the case may be.

(2) If, as in the last case, S_n increases arithmetically indefinitely with n, but however great n_k may be chosen there are both positive and negative numbers among S_{n_k}, S_{n_k+1}, ... S_{n_k+m}, ..., the series may be said to oscillate between indefinite limits of indeterminancy. It is however, in this case, usually spoken of as divergent, and its behaviour may be denoted by $LS_n=\pm\infty$.

(3) It may happen that, although S_n has no definite limit as n is indefinitely increased, it is possible to select a sequence of increasing values of n, say n_1, n_2, ... n_p, ... so that S_n converges to a definite limit provided n is restricted to have only the values in this sequence.

In this case the series is said to be an oscillating series; but oscillating series are sometimes spoken of as divergent. An oscillating series in which S_n is for every value of n numerically less than some fixed positive number is said to oscillate between finite limits of indeterminancy.

It is easily seen that if the terms of a series have all the same sign, the series is divergent in accordance with case (1), unless it is convergent.

The series $1+2+3+\ldots+n+\ldots$, $1/1+1/2+\ldots+1/n+\ldots$ are both divergent, since in each case S_n increases indefinitely with n, and is of fixed sign.

The series $1-2+3-4+5-\ldots$ oscillates between indefinite limits of indeterminancy. For $S_n=-\frac{1}{2}n$, when n is even, and $S_n=\frac{1}{2}(n+1)$, when n is odd; thus S_n increases in numerical value indefinitely as n increases, and $LS_n=\pm\infty$.

The series $1+1-2+1+1-2+1+1-2+\ldots$ oscillates between finite limits of indeterminancy. S_n has the value 1, 2, or 0 according as n is of the form $3r+1$, $3r+2$, or $3r$.

The series $\sin a+\sin 2a+\ldots+\sin na+\ldots$, where a has any fixed value which is neither zero nor a multiple of π, oscillates between finite limits of indeterminancy. In this case

$$S_n=\sin\frac{(n+1)a}{2}\sin\frac{na}{2}\operatorname{cosec}\frac{a}{2}=\frac{1}{2}\left\{\cos\frac{a}{2}-\cos\left(n+\frac{1}{2}\right)a\right\}\operatorname{cosec}\frac{a}{2}$$

It is thus seen that S_n does not converge to a definite limit, since $\cos(n+\frac{1}{2})a$ has no definite limit when n is indefinitely increased; but S_n is numerically less than, or equal to, $\frac{1}{2}\left(1+\cos\frac{a}{2}\right)\operatorname{cosec}\frac{a}{2}$, for every value of n.

193(1). *The necessary and sufficient condition for the convergence of the series* $a_1+a_2+\ldots+a_n+\ldots$ *is that, corresponding to each arbitrarily chosen positive number* η, *as small as we please, a value* n_η *of* n *can be determined, such that all the partial remainders after* n_η *terms are arithmetically less than* η

To shew that the condition is necessary, let us assume that the series is convergent, so that S exists. A value n_η of n can then be determined, such that

$$S-S_{n_\eta},\quad S-S_{n_\eta+1},\quad S-S_{n_\eta+2},\ \ldots\quad S-S_{n_\eta+m},\ \ldots$$

are all arithmetically less than $\frac{1}{2}\eta$. This is an expression of the fact that $LS_n=S$, when arbitrary values of η are taken into account.

Now

$$a_{n_\eta+1}+a_{n_\eta+2}+\ldots+a_{n_\eta+m}=(S-S_{n_\eta})-(S-S_{n_\eta+m});$$

and it then follows that, since $S-S_{n_\eta}$, $S-S_{n_\eta+m}$ are both numerically less than $\frac{1}{2}\eta$, $a_{n_\eta+1}+a_{n_\eta+2}+\ldots+a_{n_\eta+m}$ is numerically less than η; and this holds for all the values 1, 2, 3, ... of m.

Next, to shew that the condition is sufficient, we have recourse to a principle known as the General Principle of Convergence[1], in accordance with which a sequence of numbers $S_1, S_2, \dots S_n, \dots$ has a definite limit, provided that, corresponding to each arbitrarily chosen positive number η, a value n_η of n can be determined such that all the numbers

$$S_{n_\eta+1}-S_{n_\eta},\quad S_{n_\eta+2}-S_{n_\eta},\ \dots\ S_{n_\eta+m}-S_{n_\eta},\ \dots$$

are arithmetically less than η. To see the sufficiency of the condition we have then only to observe that $S_{n_\eta+m}-S_{n_\eta}$ is equal to the partial remainder $R_{n_\eta, m}$, or $a_{n_\eta+1}+a_{n_\eta+2}+\dots+a_{n_\eta+m}$.

If we take $m=1$, the condition includes that a_{n+1} may be made arbitrarily small by taking a large enough value of n; it follows that a necessary condition of convergence of the series is that $La_n=0$. This condition is however not by itself sufficient.

The rapidity of the convergence of a convergent series may be measured by the least value of n corresponding to a given value of ϵ, which is such that all the partial remainders $R_{n,m}$ are arithmetically less than ϵ; that is by the number of terms which it is necessary to take in order that the partial remainders may be all numerically less than some assigned number.

In the case of the geometrical series $1+x+x^2\dots$ which converges to the value $1/(1-x)$, when x is numerically less than unity, we see that

$$a_{n+1}+\dots+a_{n+m}=\frac{x^n(1-x^m)}{1-x},$$

and supposing x to be positive, this will be less than ϵ for all values of m, if $\frac{x^n}{1-x}<\epsilon$; in this case a suitable value of n is the integer next greater than $\frac{\log\epsilon+\log(1-x)}{\log x}$. The value of n increases as x increases, thus the rapidity of convergence of the series diminishes as x increases; when x approaches unity n increases indefinitely; thus the convergence of the series becomes indefinitely slow. When $x=1$, the series is, of course, divergent.

194. Let us next consider the case of a convergent series $a_1+a_2+\dots+a_n+\dots$ in which there are an indefinite number of positive terms and also an indefinite number of negative terms. Denoting by $|a_n|$ the numerical value of a_n, so that $|a_n|$ is equal to a_n or to $-a_n$, according as a_n is positive or negative, let us consider the series

$$|a_1|+|a_2|+|a_3|+\dots+|a_n|+\dots.$$

In case this last series is convergent the original convergent series is said to be *absolutely convergent*, whereas, if the series $\Sigma\,|a_n|$ is divergent, the series Σa_n is said to be *semi-convergent*, or *conditionally convergent*, or *accidentally convergent*.

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[1] See the author's work *On the theory of functions of a real variable*, p. 36, where this fundamental principle is discussed.

The series $1^{-2}-2^{-2}+3^{-2}+\ldots$ is absolutely convergent, since the series $1^{-2}+2^{-2}+3^{-2}+\ldots$ is convergent; but the series $1^{-1}-2^{-1}+3^{-1}-\ldots$ is only conditionally convergent, as the series $1^{-1}+2^{-1}+3^{-1}+\ldots$ is divergent.

A series $a_1-a_2+a_3-\ldots$, in which the terms are of alternate signs, is always convergent (either absolutely or conditionally) if each term is numerically greater than the next following, provided also $La_n=0$. For

$$(-1)^n R_{n,m}=(a_{n+1}-a_{n+2})+(a_{n+3}-a_{n+r})+\ldots=a_{n+1}-(a_{n+2}-a_{n+3})-\ldots,$$

hence $(-1)^n R_{n,m}$ is positive and less than or equal to a_{n+1}. It follows that n may be chosen so great that $|R_{n,m}|<\epsilon$, for all values of m, however small ϵ may be chosen; and thus the series is convergent.

195. In a conditionally convergent series the order of the terms cannot in general be deranged without altering the sum. Let S_p be the sum of the first p positive terms, and S'_q the sum of the first q negative terms with their signs changed, then if the series be re-arranged so that the sequence of the positive terms is unaltered, and also that of the negative terms, but so that of the first $p+q$ terms, p are positive and q are negative, the sum of the series so re-arranged is the limit of $S_p-S'_q$, when p and q are indefinitely increased. Now the two series S_p, S'_q each consists of positive terms, hence the limits of S_p and of S'_q are each either finite and definite or else infinite; by hypothesis they are not both finite and definite, as the given series is not absolutely convergent, hence one at least of the limits S_p, S'_q is infinite; if both are infinite the value of $L(S_p-S'_q)$ will depend on the two sequences of values of p and q. If one only of the limits S_p, S'_q is infinite, $L(S_p-S'_q)$ is infinite and the original series was not convergent. If in the original order $a_1-a_2+a_3\ldots$ of the series the signs are alternately positive and negative, p and q become indefinitely great in a ratio of equality, but if, for example, we write the series $a_1+a_3-a_2+a_5+a_7-a_4+\ldots$, p and q become indefinitely great in the ratio 2 : 1, and the limits of $S_{2q}-S'_q$, and $S_q-S'_q$ when q is indefinitely increased, are in general not equal.

As an example, consider the semi-convergent series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$; denote its sum by S, then

$$S_{4n}=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots-\frac{1}{4n}$$

$$=\sum_1^n\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right).$$

Let S' denote the sum of the series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\dots$ in which the order of terms in the series S has been altered, we have

$$S'_{3n}=\sum_1^n\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right),$$

hence

$$S'_{3n}-S_{4n}=\sum_1^n\left(\frac{1}{4n-2}-\frac{1}{4n}\right)$$

$$=\tfrac{1}{2}\sum_1^n\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\tfrac{1}{2}S_{2n},$$

when n becomes indefinitely great, we have therefore $S'=\frac{3}{2}S$. This example was given by Dirichlet, who first pointed out that the sum of a semi-convergent series depends on the order of the terms.

196. *Riemann* has shewn that the terms in a semi-convergent series may be so re-arranged that the limiting sum of the new series may have any given value α.

Suppose α is positive; take first p positive terms, p being such that $S_{p-1}<\alpha$ and $S_p>\alpha$; then take q negative terms, q being so chosen that $S_p-S'_{q-1}>\alpha$, and $S_p-S'_q<\alpha$; next take p' positive terms such that $S_{p+p'-1}-S'_q<\alpha$, and $S_{p+p'}-S'_q>\alpha$, then q' negative terms such that $S_{p+p'}-S'_{q+q'}<\alpha$, and $S_{p+p'}-S'_{q+q'-1}>\alpha$, and so on. Proceeding in this way, we obtain a series such that its sum differs from α by less than its last term, hence when we make the number of terms indefinitely great its sum will converge to α.

It can also be shewn that the terms may be so re-arranged that the new series diverges, or that it oscillates.

The convergence of complex series.

197. Suppose $z_1, z_2, z_3, \dots z_n, \dots$ to be a sequence of complex numbers; thus z_n denotes x_n+iy_n, where x_n and y_n are real numbers. Let

$$S_n=z_1+z_2+\dots+z_n,\quad s_n=x_1+x_2+\dots+x_n,\quad s'_n=y_1+y_2+\dots+y_n;$$

thus $S_n=s_n+is'_n$.

If S_n has a definite limit S, itself a complex or real number, when n is indefinitely increased, the infinite series

$$z_1+z_2+\dots+z_n+\dots$$

is said to be convergent, and S is called its limiting sum, or simply its sum.

The condition that $S=LS_n$ is that $|S-S_n|$ converges to zero as n is indefinitely increased; thus if

$$S-S_n=\rho_n(\cos\theta_n+i\sin\theta_n),$$

we must have $L\rho_n = 0$. If $S = s + is'$, when s and s' are real, we have $s - s_n = \rho_n \cos\theta_n$, $s' - s'_n = \rho_n \sin\theta_n$; it then follows that, if $L\rho_n = 0$, we also have $L(s - s_n) = 0$, $L(s' - s'_n) = 0$, or s_n, s'_n converge to s and s' respectively. It thus appears that in order that the series $z_1 + z_2 + z_3 + \ldots$ may be convergent, it is necessary that the two series $x_1 + x_2 + x_3 + \ldots$, $y_1 + y_2 + y_3 + \ldots$ should both be convergent. Conversely if these latter series are convergent, the series of complex numbers is also convergent, for

$$|(s+is') - (s_n + is'_n)| \leqq |s - s_n| + |s' - s'_n|;$$

if now $Ls_n = s$, $Ls'_n = s'$, we can choose a value n_ϵ of n so large that $|s - s_n| < \frac{1}{2}\epsilon$, $|s' - s'_n| < \frac{1}{2}\epsilon$, provided $n \geqq n_\epsilon$. It follows that $|(s+is') - (s_n + is'_n)| \leqq \epsilon$, if $n \geqq n_\epsilon$; and since ϵ is arbitrary we therefore have $L(s_n + is'_n) = s + is'$, and thus the series of complex numbers is convergent. In case the limiting value of either of the sums Σx, Σy is not finite, or in case either of these series oscillates, the series Σz is not convergent.

Suppose $z_n = r_n(\cos\theta_n + i\sin\theta_n)$, then we shall shew that the series Σz is convergent provided the series Σr, in which each term r_n is the modulus of the corresponding term z_n, is convergent. The given series $\Sigma r_n(\cos\theta_n + i\sin\theta_n)$ is convergent provided each of the series $\Sigma r_n \cos\theta_n$, $\Sigma r_n \sin\theta_n$ is convergent; now each of the numbers $r_n \cos\theta_n$, $r_n \sin\theta_n$ lies between the numbers $\pm r_n$; also the number $S_{n+m} - S_n$ is for either of the series $\Sigma r\cos\theta$, $\Sigma r\sin\theta$ numerically less than the corresponding partial remainder for the series Σr. If then the latter series is convergent, so is each of the former ones; hence the series Σz_n is convergent.

The converse is not necessarily true; thus the series

$$\Sigma r_n(\cos\theta_n + i\sin\theta_n)$$

may be convergent, whilst Σr_n is divergent.

If the series Σr_n formed by the sum of the moduli is convergent, then the series $\Sigma r_n(\cos\theta_n + i\sin\theta_n)$ is said to be *absolutely convergent*.

For example, the series of which the general term is $n^{-2}(\cos n\theta + i\sin n\theta)$ is absolutely convergent, since the series Σn^{-2} converges, whereas the convergent series of which the general term is $n^{-1}(\cos n\theta + i\sin n\theta)$, $(2\pi > \theta > 0)$, is not absolutely convergent, since the series Σn^{-1} is divergent.

Continuous functions.

198. Suppose $f(z)$ to be a function of the complex variable $z=x+iy$, which has a single finite value for every value of z which lies within any given limits; this function will then have a single value for every point in the diagram, which lies within a certain area; this area may be any finite portion of the plane of representation of z, or the whole of that plane.

Such a function is said to be *continuous* at the point $z=z_1$, *if a positive number* η *can always be found such that the modulus of* $f(z)-f(z_1)$ *is less than an assigned positive number* ϵ, *taken as small as we please, for all values of* z *which are such that the modulus of* $z-z_1$ *is less than* η. *For each value of* ϵ *a value of* η *must exist.*

A function which satisfies this condition at every point within any given area, is said to be continuous in that area. The boundary of the area may, or may not, be included.

Uniform convergence.

199. Let $f_n(z)$ be a function of z or $x+iy$, which is continuous in any area; then if the series

$$f_1(z)+f_2(z)+\ldots+f_n(z)+\ldots$$

is convergent, we may denote its limiting sum by $F(z)$. Suppose

$$f_1(z)+f_2(z)+\ldots+f_n(z),$$

where n is any fixed number, is equal to $S_n(z)$, then the limiting sum of $f_{n+1}(z)+f_{n+2}(z)+\ldots$ is called the remainder after n terms, and may be denoted by $R_n(z)$; we have therefore

$$F(z)=S_n(z)+R_n(z).$$

Now suppose that, corresponding to any given positive number ϵ, however small, a value of n, *independent of* z, can be found, such that for all values of z represented by points lying in any given area, the modulus of $R_m(z)$ is less than ϵ, where m is equal to *or greater than* n, the series is said to *converge uniformly* for all values of z represented by points in that area. The integer n will depend in value upon ϵ.

If as z approaches indefinitely near any fixed value z_1 in the area, in order that the moduli of all the remainders $R_m(z)$ may be less than ϵ, it is necessary to suppose n to increase indefinitely, then in the neighbourhood of the point z_1, the series does not converge uniformly and is said to *converge infinitely slowly.* A point z_1 for which ϵ can be so chosen that this happens is said to be a point in the neighbourhood of which the convergence is non-uniform, or sometimes simply a point of non-uniform convergence in case the series converges at that point itself. For any space including such a point it is impossible to assign any fixed value of n, such that for all values of z within that space, the moduli of R_m are less than the sufficiently small positive number ϵ; and thus the series does not converge uniformly throughout that space. When z is equal to z_1, the series may be either convergent or divergent.

We may state the matter as follows:

Suppose that as z approaches some fixed value z_1 a positive number ϵ can be assigned such that the number of terms n of the series $f_1(z)+f_2(z)+\dots$ which must be taken, in order that mod. $R_m(z)<\epsilon$, where m is equal to or greater than n, depends on the modulus of $z-z_1$ in such a way that n continually increases as mod. $(z-z_1)$ diminishes, and becomes indefinitely great when mod. $(z-z_1)$ becomes indefinitely small, the series is said to converge non-uniformly in the neighbourhood of z_1.

In the neighbourhood of such a point, the rate of convergence of the series varies infinitely rapidly, and when mod. $(z-z_1)$ is diminished indefinitely, the series converges indefinitely slowly.

It should be observed that a convergent *numerical* series cannot converge infinitely slowly; thus when z is equal to z_1, the convergence of the series $f_1(z_1)+f_2(z_1)+\dots$, if it is convergent, is not indefinitely slow; it is only when z is a variable such that mod. $(z-z_1)$ is indefinitely diminished, that the series

$$f_1(z)+f_2(z)+\dots$$

converges infinitely slowly. It is consequently more exact to speak of the non-uniform convergence of a series in the neighbourhood of a point, than at the point itself. The number of terms n that must be taken in order that the modulus of the remainder $R_n(z)$ may be less than the sufficiently small number ϵ, increases as z approaches the value z_1, becomes indefinitely great when mod. $(z-z_1)$ becomes continually smaller, and then, if the series is convergent at the point z_1, suddenly changes to a finite value; this number n is therefore itself discontinuous in the neighbourhood of such a point.

If in any area A we have, at every point of the area,

$$|f_1(z)| \leqq a_1, |f_2(z)| \leqq a_2, \dots |f_n(z)| \leqq a_n, \dots,$$

where $a_1, a_2, \dots a_n, \dots$ are fixed positive numbers such that the series $a_1 + a_2 + \dots + a_n + \dots$ is convergent, then the series

$$f_1(z) + f_2(z) + \dots$$

is uniformly convergent in the area A. This theorem affords a test of uniform convergence which is of great value in application to particular cases; it is known as Weierstrass's test. To prove it, we observe that, if ϵ be an arbitrarily chosen positive number, n_ϵ may be so chosen that $a_{n+1} + a_{n+2} + \dots + a_{n+m}$ is, for every value of m, less than ϵ, where $n \geqq n_\epsilon$. The modulus of

$$|f_{n+1}(z) + f_{n+2}(z) + \dots + f_{n+m}(z)|$$

is, for every value of z, not greater than $a_{n+1} + a_{n+2} + \dots + a_{n+m}$, and is therefore less than ϵ. Since this holds for every value of m, we see that the complex series is convergent, and that for every value of z, $|R_n(z)| < \epsilon$, provided $n \geqq n_\epsilon$. Therefore the series converges uniformly in A.

By some writers, a series is defined to be uniformly convergent in a given area, when a number n can be found such that for all values of z, the modulus of the remainder R_n is less than ϵ. The definition given in the text is more stringent than the one here mentioned; it is possible to construct series which converge uniformly according to the latter but not according to the former definition.

200. If the functions $f_1(z)$, $f_2(z)$, ... are continuous for all values of z represented by points lying in a given area A, then the function $F(z)$ which represents the sum of a convergent series $\Sigma f(z)$, is a continuous function for all values of z represented by points lying in the area A, *provided the series* $\Sigma f(z)$ *converges uniformly in the whole area* A.

For we have $F(z) = S_n + R_n$, n being such that for all values of z to be considered, the modulus of R_n is less than ϵ; let z receive an increment δz, and let $\delta F(z)$, δS_n, δR_n be the corresponding increments of $F(z)$, S_n, and R_n. Then, since by supposition the moduli of R_n and $R_n + \delta R_n$ are both less than ϵ, the modulus of δR_n is less than 2ϵ. Also since S_n is a continuous function of z, if the modulus of δz be small enough, the modulus of δS_n is less than ϵ; hence, provided mod. δz is less than a certain value, the modulus of $\delta S_n + \delta R_n$ or of $\delta F(z)$ is less than 3ϵ, since the modulus of $\delta S_n + \delta R_n$ is not greater than the sum of the moduli of δS_n and δR_n. Now 3ϵ can be made as small as we please, therefore mod. $\delta F(z)$ can be made as small as we please by making mod. δz small enough; that is to say the function $F(z)$ is continuous.

It will be observed that for this proof, the less stringent definition of uniform convergence, given in the note to Art. 199, is sufficient.

201. For a value z_1 of z, for which the series converges non-uniformly in the neighbourhood, the sum of the series is not necessarily continuous; in this case the reasoning of the last Article fails. The limiting value of the function $f_n(z)$, when $z = z_1$, is $f_n(z_1)$, but it does not follow that $\sum\limits_1^\infty \{f_n(z) - f_n(z_1)\}$ converges to zero as z converges to z_1. We may denote the sum $\sum\limits_1^n \{f(z) - f(z_1)\}$ by $F(n, z - z_1)$, a function of n, and of $z - z_1$; now the limiting value of $F(n, z - z_1)$ when z is first made equal to z_1, and then n is afterwards made infinite, is zero; but if n is first made infinite, and afterwards $z - z_1$ is made zero, the limiting value of $F(n, z - z_1)$ is not necessarily zero.

As an example of this phenomenon, Stokes considers the real series

$$\frac{1+5x}{2(1+x)} + \dots + \frac{x(x+2)n^2 + x(4-x)n + 1 - x}{n(n+1)\{(n-1)x+1\}(nx+1)} + \dots;$$

when $x=0$, this series becomes

$$\frac{1}{1\,.\,2} + \dots + \frac{1}{n(n+1)} + \dots.$$

Now the general term is

$$\frac{1}{n(n+1)} + \frac{2x}{\{(n-1)x+1\}(nx+1)},$$

or

$$\left\{\frac{1}{n} + \frac{2}{(n-1)x+1}\right\} - \left\{\frac{1}{n+1} + \frac{2}{nx+1}\right\},$$

therefore the sum of the series is 3, whatever value different from zero x may have; the sum of the series $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \dots$ is however unity, and thus the sum of the series is discontinuous in the neighbourhood of the value of $x=0$.

The remainder after n terms is $\frac{1}{n+1} + \frac{2}{nx+1}$; putting this equal to ϵ, we find

$$n = \{x + 2 - \epsilon(x+1) + \sqrt{\{\epsilon(x+1) - (x+2)\}^2 - 4\epsilon x(\epsilon - 3)}\}/2\epsilon x,$$

which increases indefinitely as x becomes indefinitely small; thus the series converges infinitely slowly when x is infinitely small; this is the reason of the discontinuity in the sum of the series.

The discovery of the distinction between uniform and non-uniform convergence of series has usually been attributed to *Seidel*, who published his "Note über eine Eigenschaft der Reihen welche discontinuirliche Functionen darstellen" in the *Transactions* of the Bavarian Academy for 1848; the theory had, however, been previously published by *Stokes*, in a paper "On the Critical Values of the sums of Periodic Series[1]," read on Dec. 6, 1847, before the Cambridge Philosophical Society. Although the theory is in some respects stated more fully by Seidel than by Stokes, the latter must be considered to have the priority in the discovery of the true cause of discontinuity in the functions represented by infinite series[2]. The distinction between uniform and non-uniform convergence has played a very important part in the modern developments of the subject.

The matter is summed up by Seidel in the following theorem:—Having given a convergent series, of which the single terms are continuous functions of a variable z, and which represents a discontinuous function of z: one must be able, in the immediate neighbourhood of a point where the function is discontinuous, to assign values of z for which the series converges with any arbitrary degree of slowness.

The geometrical series.

202. Consider the geometrical series $1 + z + z^2 + \dots + z^{n-1}$, where $z = x + iy = r(\cos\theta + i\sin\theta)$. We have for the sum of this series the value

$$\frac{1 - z^n}{1 - z} \text{ or } \frac{1 - r^n(\cos n\theta + i\sin n\theta)}{1 - r(\cos\theta + i\sin\theta)};$$

put $$1 - r\cos\theta = \rho\cos\phi, \quad r\sin\theta = \rho\sin\phi,$$

then $$\rho = +\sqrt{1 - 2r\cos\theta + r^2},$$

the sum then becomes

$$\frac{1}{\rho}(\cos\phi + i\sin\phi) - \frac{r^n}{\rho}\left\{\cos(n\theta + \phi) + i\sin(n\theta + \phi)\right\};$$

and when n is made indefinitely great, the modulus of the second term in this sum becomes indefinitely small, if $r < 1$; but if $r > 1$, it becomes infinite. Thus the infinite series

$$1 + z + z^2 + \dots + z^{n-1} + \dots$$

converges if the modulus of z is less than unity, and its sum is then

$$\frac{1}{\rho}(\cos\phi + i\sin\phi) = \frac{1 - r\cos\theta + i\,.\,r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

If the modulus of z is greater than unity, the series is divergent;

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[1] See Stokes' *Collected Works*, Vol. I.

[2] On the history of this discovery see Reiff's *Geschichte der unendlichen Reihen.*

and if mod. z is unity it is also not convergent, since the sums of the two series $\Sigma \cos n\theta$, $\Sigma \sin n\theta$, which have been found in Art. 74, do not approach a definite value when n is indefinitely great.

We have, by equating the real and imaginary parts of the series and the sum,

$$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = 1 + r\cos\theta + r^2\cos 2\theta + \ldots + r^n \cos n\theta + \ldots,$$

$$\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} = r\sin\theta + r^2\sin 2\theta + \ldots + r^n \sin n\theta + \ldots;$$

these series hold for all values of r lying between ± 1, excluding $r = 1$ and $r = -1$, for which the series are not convergent. To see that this is the case, we need only write $-z$ for z in the original series.

The geometrical series is uniformly convergent for all values of z of which the modulus is $\leqq 1 - \eta$, where η is any fixed positive number, arbitrarily small. For the remainder after the first n terms is $\dfrac{z^n}{1 - z}$, and the modulus of this less than $\dfrac{(1-\eta)^n}{\eta}$; the series will then be such that for all values of z of which the modulus is $\leqq 1 - \eta$, $|R_n(z)| < \epsilon$, if

$$\frac{(1-\eta)^n}{\eta} < \epsilon, \text{ or if } n > \frac{\log\eta + \log\epsilon}{\log(1-\eta)}.$$

Hence, since it is possible to choose n so that for all values of z of which the moduli are $\leqq 1 - \eta$, the remainders after n terms are less than ϵ, and since this clearly holds for all greater values of n, the series converges uniformly for all such values.

It has thus been shewn that the geometrical series is uniformly convergent in the area bounded by any circle concentric with and interior to the circle of radius unity with the centre at the origin.

Series of ascending integral powers.

203. We shall now consider the general power-series

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots,$$

where $a_0, a_1, a_2, \ldots$ are complex numbers independent of the complex variable z. Let r denote the modulus of z, and $\alpha_0, \alpha_1, \alpha_2, \ldots$ the moduli of $a_0, a_1, a_2, \ldots$. The series of moduli is

$$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \ldots + \alpha_n r^n + \ldots;$$

when this series is convergent the series in powers of z is absolutely convergent. If the series of moduli converges for any value of r it is convergent for all smaller values of r; and if it is divergent for any value of r it is also divergent for all greater values of r. As regards this series $a_0 + a_1 r + a_2 r^2 + \dots$, three cases may arise.

(1) The series may converge for some values of r different from zero, and diverge for other values; there then exists a positive number ρ such that the series converges when $r < \rho$, and diverges when $r > \rho$. When $r = \rho$ the series may either converge or diverge, as the case may be.

(2) The series may converge for all values of r; it is convenient to express this by $\rho = \infty$.

(3) The series may diverge for all values of r except $r = 0$; this may be expressed by $\rho = 0$.

In order to determine the number ρ in any given case, we consider the values of $a_n^{\frac{1}{n}}$. It may happen that, as n is indefinitely increased, $a_n^{\frac{1}{n}}$ converges to a definite limit A; in that case, if ϵ be an arbitrarily chosen positive number, as small as we please, $a_n^{\frac{1}{n}}$ lies between $A + \epsilon$ and $A - \epsilon$ for all values of n with the exception of a finite number of such values. More generally, it may happen that a positive number A exists, such that, for all values of n except a finite set, $a_n^{\frac{1}{n}} < A + \epsilon$, and such that for an infinite number of values of n, $a_n^{\frac{1}{n}}$ lies between $A + \epsilon$ and $A - \epsilon$. In either case, the number ρ is equal to $1/A$. To see this it will be sufficient to prove that the series converges if $r < 1/A$, and that it diverges if $r > 1/A$. For all values of n except a finite set $a_n r^n < (A + \epsilon)^n r^n$, where ϵ may be arbitrarily chosen; if r has a value $< 1/A$, we can choose ϵ so that $(A + \epsilon) r < 1$. All the terms of the series, except a finite set of them, are then less than the corresponding terms of the geometric series of which the common ratio $(A + \epsilon) r$ is less than unity; consequently the series is convergent. If $r > 1/A$, we can choose ϵ so that $(A - \epsilon) r > 1$, and thus $a_n r^n > (A - \epsilon)^n r^n > 1$, for an infinite number of values of n; the series is consequently divergent.

If $a_n^{\frac{1}{n}}$ converges to the limit zero, as n is indefinitely increased, the series converges for every value of r. For, in that case,

$a_n r^n < \epsilon^n r^n$, where ϵ may be so chosen that $\epsilon r < 1$; and this holds for every value of n except a finite set of such values. Each term of the series, with the exception of a finite number, is then less than the corresponding term of a convergent geometric series; consequently the series is convergent. In this case $\rho = \infty$.

If $a_n^{\frac{1}{n}}$ has indefinitely great values, that is, if no number exists which is greater than all the numbers $a_n^{\frac{1}{n}}$, the series diverges for all values of r except $r = 0$. In this case $\rho = 0$. For, if r have any given value except zero, there are an infinite number of terms of the series each of which is greater than unity, and thus the series is divergent.

204. In the last Article it has been shewn that a number ρ exists, which may however be zero, or may have the improper value ∞, which is such that the series $a_0 + a_1 r + a_2 r^2 + \dots$ is convergent for each value of r which is $< \rho$, and is divergent for each value of r that is $> \rho$.

About the point $z = 0$ as centre, describe a circle of radius ρ; this circle is called the *circle of convergence* of the series

$$a_0 + a_1 z + a_2 z^2 + \dots,$$

and its radius is called the *radius of convergence* of the series.

The radius of convergence may be finite, or zero, or infinite.

It will be shewn that the series $a_0 + a_1 z + a_2 z^2 + \dots$ is absolutely convergent for any point z in the interior of the circle of convergence, and that it is divergent for any point z exterior to that circle. No quite general statement can be made as regards the convergence of the series for a point on the circumference of the circle of convergence.

That the series is absolutely convergent if mod. $z < \rho$ follows from the fact that the series of moduli is then convergent. That the series is divergent if mod. z has a value $r > \rho$ is seen from the fact that the necessary condition of convergence $L\,|a_n z^n| = 0$ is not satisfied. For $|a_n z^n| = (r/\rho)^n a_n \rho^n$; and for an infinite number of values of n, $a_n \rho^n > (1 - \rho\epsilon)^n$; hence if ϵ be so chosen that

$$r\left(\frac{1}{\rho} - \epsilon\right) > 1,$$

we see that $|a_n z^n| > 1$, for an infinite number of values of n.

205. It will next be shewn that the series $a_0 + a_1 z + a_2 z^2 + \ldots$ converges uniformly in any circle of which the radius is less than the radius of convergence, and of which $z = 0$ is the centre. Suppose $\rho - k$ to be the radius of this circle, and let ρ_1 be a fixed number between ρ and $\rho - k$; let $\rho - k = \rho_1 - h$. The modulus of the limiting sum of $a_n z^n + a_{n+1} z^{n+1} + \ldots$ does not exceed the limiting sum of the series

$$\alpha_n r^n + \alpha_{n+1} r^{n+1} + \ldots,$$

or
$$\alpha_n \rho_1^n (r/\rho_1)^n + \alpha_{n+1} \rho_1^{n+1} (r/\rho_1)^{n+1} + \ldots.$$

Now the numbers $\alpha_n \rho_1^n$, $\alpha_{n+1} \rho_1^{n+1}, \ldots$ are all less than some fixed number K, since the series is convergent when $r = \rho_1$; thus the sum of the series is less than $K\{(r/\rho_1)^n + (r/\rho_1)^{n+1} + \ldots\}$, or than $K(r/\rho_1)^n (1 - r/\rho_1)^{-1}$; and this is less than $K(1 - h/\rho_1)^n \rho_1/h$. If ϵ be an arbitrarily chosen positive number, a value n_1 of n can be determined such that $K(1 - h/\rho_1)^n \rho_1/h < \epsilon$, for $n \geqq n_1$. Hence the modulus of the remainder $R_n(z)$ of the series $a_0 + a_1 z + a_2 z^2 + \ldots$ is less than ϵ, for $n \geqq n_1$, and for all values of z such that mod. $z \leqq \rho - k$; therefore the convergence of the series is uniform in the circle of radius $\rho - k$. This is true however small the number $k (> 0)$ may be taken to be, but it would be incorrect to assert that the convergence is necessarily uniform in the circle of convergence.

Denoting by $F(z)$ the sum of the series $a_0 + a_1 z + a_2 z^2 + \ldots$ for values of z of which the moduli are less than the radius of convergence, it follows from Art. 200 that $F(z)$ is a continuous function of z, for all points lying in the interior of the circle of convergence. If the radius of convergence is infinite, $F(z)$ is continuous for all finite points in the plane.

The series
$$1 + z + z^2 + z^3 + \ldots,$$
$$1 + \frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$
have the radius of convergence unity; their sum-functions are continuous functions of z in the interior of the circle of radius unity.

The series
$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$
has the radius of convergence infinite; the sum-function is continuous for all finite values of z.

The series
$$1 + 1!z + 2!z^2 + \ldots + n!z^n + \ldots$$
has the radius of convergence zero.

206. The convergence of the series on the circle of convergence itself has not yet been considered; we may without loss of generality take the radius of convergence to be unity.

It can be shewn that the series $a_0 + a_1 z + a_2 z^2 + \dots$, when the coefficients are real, converges for points on the circle of convergence, with the exception of the point $z = 1$, if the coefficients are all positive, and with the exception of the point $z = -1$, when the coefficients are alternately positive and negative, provided in both cases the coefficients $a_0, a_1, a_2, \dots$ are in descending order of absolute magnitude, and provided the limit of a_n, when n is indefinitely increased, is zero.

Let $$S_n = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$$

and suppose the coefficients all positive, then

$$S_n(1 - z) = a_0 - a_{n-1} z^n - z\{(a_0 - a_1) + (a_1 - a_2) z + (a_2 - a_3) z^2 + \dots$$
$$+ (a_{n-2} - a_{n-1}) z^{n-2}\};$$

now the series

$$(a_0 - a_1) + (a_1 - a_2) + (a_2 - a_3) + \dots$$

is convergent (see Art. 194, note), therefore the two series

$$(a_0 - a_1) + (a_1 - a_2) \cos \theta + (a_2 - a_3) \cos 2\theta + \dots,$$

$$(a_0 - a_1) + (a_1 - a_2) \sin \theta + (a_2 - a_3) \sin 2\theta + \dots$$

are also convergent, since the cosines and sines all lie between ± 1, thus the series

$$(a_0 - a_1) + (a_1 - a_2) z + (a_2 - a_3) z^2 + \dots$$

is convergent when mod. $z = 1$; since $|a_{n-1} z^n|$ has the limit zero when n is infinite, we see that $LS_n(1 - z)$ is finite when mod. $z = 1$; hence unless $z = 1$, LS_n is finite.

If the coefficients in the series are of alternate signs, change z into $-z$, then this case is reduced to the last.

Whether the series is convergent when $z = 1$, or in the case of coefficients of alternate signs, when $z = -1$, has not been determined, and depends upon the particular series. The series may be only semi-convergent on the circle of convergence.

If the coefficients of the series are complex, we can divide the series into two, in one of which the coefficients are real and in the other imaginary; the two series can then be considered separately.

The series $$1+\frac{z}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\ldots$$ is convergent when mod. $z=1$, except when $z=1$. Thus the two series $\Sigma\frac{1}{n}\cos n\theta$, $\Sigma\frac{1}{n}\sin n\theta$ are both convergent, except that the first is divergent when θ is zero or an even multiple of π.

207. Suppose $F(x)$ is the continuous function of x which is represented as the sum of the series $a_0+a_1x+a_2x^2+\ldots$, with real coefficients, which converges for real values of x, less than unity. Let us assume that the series diverges when $x>1$, but that the series $a_0+a_1+a_2+\ldots$, corresponding to $x=1$, is convergent.

It will then be shewn that the sum of the series $a_0+a_1+a_2+\ldots$ is the limit of $F(x)$ when x increases from values less than unity to unity as its limit. Thus the continuous function $F(x)$ defined for $x=1$ by $F(1)=\underset{x=1}{L}F(x)$ continues to represent the sum of the series when $x=1$. This theorem was given by Abel[1].

Let $s_n=a_0+a_1+a_2+\ldots+a_n$; $s_0=a_0$. In virtue of a theorem which will be proved in Art. 209, since the two series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$
$$1+x+x^2+\ldots+x^n+\ldots$$

are both absolutely convergent when $x<1$, their product, formed by multiplication,

$$s_0+s_1x+s_2x^2+\ldots+s_nx^n+\ldots$$

is convergent, and its limiting sum is $F(x)/(1-x)$, the product of the limiting sums of the two series. Denoting Ls_n by s, the number n can be chosen such that $s_n, s_{n+1}, s_{n+2}, \ldots$ all lie between $s+\epsilon$ and $s-\epsilon$, where ϵ is an arbitrarily chosen positive number.

The limiting sum of $s_nx^n+s_{n+1}x^{n+1}+\ldots$, for such a value of n, lies between $(s+\epsilon)x^n/(1-x)$ and $(s-\epsilon)x^n/(1-x)$. Therefore $F(x)$ lies between

$$(s+\epsilon)x^n+(1-x)(s_0+s_1x+\ldots+s_{n-1}x^{n-1})$$

and $$(s-\epsilon)x^n+(1-x)(s_0+s_1x+\ldots+s_{n-1}x^{n-1}).$$

It follows that $|F(x)-s|$ is less than

$$\epsilon+|s|(1-x^n)+(1-x)(|s_0|+|s_1|+\ldots+|s_{n-1}|).$$

The number n having been fixed, corresponding to ϵ, we can choose a value of x, say x_1, such that $|F(x)-s|$ is numerically less than 2ϵ, for $1>x\geqq x_1$, since $1-x$ and $1-x^n$ may be taken as small as we please by properly choosing x. Since 2ϵ is an arbitrarily small number, it follows that s is the limit of $F(x)$ for $x = 1$.

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[1] See *Crelle's Journal*, Vol. I.

If $a_0, a_1, a_2, \ldots$ are complex numbers, we may divide the series into two parts, one real and the other imaginary, and the theorem applies to each part separately; hence it holds for the whole series.

Next let $F(z)$ be the continuous function which represents, when mod. $z < 1$, the sum of the series $a_0 + a_1 z + a_2 z^2 + \ldots$, where z is the complex number $r(\cos\theta + i\sin\theta)$. The series may be divided into the two parts

$$a_0 + a_1 r\cos\theta + a_2 r^2 \cos 2\theta + \ldots$$
$$i(a_1 r\sin\theta + a_2 r^2 \sin 2\theta + \ldots),$$

and the theorem holds for each of these two series. Therefore if the series $a_0 + a_1 z + a_2 z^2 + \ldots$ is convergent when $z = \cos\theta + i\sin\theta$, its sum is the limit for $r = 1$, of $F(z)$, the value of θ being kept constant. The function represented by the series is then continuous at the point on the circle of convergence with the values on the radius of the circle of convergence through the point.

In order that the necessity for the investigation in this Article may be seen, we remark that a similar theorem would not hold for the series obtained by altering the order of the terms in the series $a_0 + a_1 x + a_2 x^2 + \ldots$ For example, consider the two real series

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \text{ and } x + \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{4}x^4 + \ldots;$$

as long as $x < 1$, the series are absolutely convergent, and they have the same sum; when however $x = 1$, the sums of the series are not equal, as has been shewn in Art. 195. The sum of the first series is continuous up to the value $x = 1$, of x, but that of the second is not so.

208. There cannot be two distinct series of powers of z,

$$a_0 + a_1 z + a_2 z^2 + \ldots,$$
$$b_0 + b_1 z + b_2 z^2 + \ldots,$$

which both converge to the same value $F(z)$ for all points in a circle of radius $k\,(> 0)$. For since they converge to the same value for $z = 0$, we must have $a_0 = b_0$; and thus the series $a_1 z + a_2 z^2 + \ldots$, $b_1 z + b_2 z^2 + \ldots$ converge to the same value when mod. $z \leqq k$. This is impossible unless the two series

$$a_1 + a_2 z + a_3 z^2 + \ldots, \quad b_1 + b_2 z + b_3 z^2 + \ldots$$

are both convergent and have the same limiting sums for $<$ mod. $z \leqq k$. The radii of convergence of these two series are each $\geqq k$, and their sum-functions are both continuous within their circles of convergence. Since their sum-functions are identical for each value of z except $z=0$, in the circle of radius k, it follows from the continuity of those functions that they are identical when $z=0$; therefore $a_1=b_1$. By proceeding in this manner, it can be shewn that all the corresponding coefficients in the two series are equal, and thus that the series are identical.

Convergence of the product of two series.

209. Let S, S' denote the limiting sums of two absolutely convergent series

$$a_1+a_2+a_3+\ldots+a_n+\ldots,$$
$$b_1+b_2+b_3+\ldots+b_n+\ldots;$$

then it can be shewn that the series

$$a_1b_1+(a_1b_2+a_2b_1)+\ldots+(a_1b_n+a_2b_{n-1}+\ldots+a_nb_1)+\ldots$$

obtained by multiplying together the given series is convergent, and that its limiting sum is SS'

Denote by s_n the sum of n terms of the product series, and let α, β be the moduli of a and b respectively. Since the series S, S' are absolutely convergent, the series of moduli are convergent; denote their sums by Σ, Σ', and let

$$\sigma_n=\alpha_1\beta_1+(\alpha_1\beta_2+\alpha_2\beta_1)+\ldots+(\alpha_1\beta_n+\alpha_2\beta_{n-1}+\ldots+\alpha_n\beta_1).$$

We have $S_nS_n'-s_n=a_2b_n+a_3b_{n-1}+\ldots+a_nb_n$;

hence
$$\text{mod.}\,(S_nS_n'-s_n)\leqq\alpha_2\beta_n+\alpha_3\beta_{n-1}+\ldots+\alpha_n\beta_n$$
$$\leqq\Sigma_n\Sigma_n'-\sigma_n$$

Now $\sigma_n<\Sigma_n\Sigma_n'<\sigma_{2n}$, because σ_{2n} contains more terms than the product $\Sigma_n\Sigma_n'$, whereas σ_n contains fewer; hence the limit of σ_n, when n is indefinitely increased, is finite, and therefore since the limits of σ_n, σ_{2n} must be the same, each is equal to $\Sigma\Sigma'$; thus the limit of mod. $(S_nS_n'-s_n)$ is zero, or $s=SS'$.

More generally it can be shewn that it is sufficient for the validity of the theorem that the convergence of one only of the series $a_1+a_2+\ldots$, $b_1+b_2+\ldots$ should be absolute, that of the other being conditional. In case the two series are both only conditionally convergent, the product series $a_1b_1+(a_1b_2+a_2b_1)+\ldots$ is not necessarily convergent, but it can be shewn that in case it be convergent, its sum is the product of the sums of the two given series[1].

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[1] For proofs of these results, see the author's *Theory of functions of a real variable*, pp. 500, 501.

The convergence of double series.

210. Let us consider a double sequence of positive real numbers $a_{r,s}$

$$\begin{array}{l} a_{1,1},\ a_{1,2},\ a_{1,3},\ \dots\ a_{1,s},\ \dots \\ a_{2,1},\ a_{2,2},\ a_{2,3},\ \dots\ a_{2,s},\ \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r,1},\ a_{r,2},\ \dots\dots\dots\ a_{r,s},\ \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

Let us assume that the numbers in each row when added together have a definite limiting sum; and let $s_1, s_2, \dots s_r, \dots$ denote the values of this limiting sum for the first, second, ... rth rows. Let it be further assumed that the series $s_1 + s_2 + \dots + s_r + \dots$ is convergent, and has S for its limiting sum. It will be shewn that the series $a_{1,s} + a_{2,s} + \dots + a_{r,s} + \dots$ obtained by adding the numbers in any one column is convergent, and that if its limiting sum be denoted by σ_s, the series $\sigma_1 + \sigma_2 + \sigma_3 + \dots$ is convergent, and has S for its limiting sum.

That $a_{1,s} + a_{2,s} + \dots + a_{r,s} + \dots$ is convergent follows from the fact that each term is less than the corresponding term of the convergent series $s_1 + s_2 + \dots + s_r + \dots$. An integer p may be so chosen that the r numbers

$$\sigma_1 - \sum_{n=1}^{n=p} a_{n,1},\ \sigma_2 - \sum_{n=1}^{n=p} a_{n,2},\ \dots\ \sigma_r - \sum_{n=1}^{n=p} a_{n,r}$$

are all less than ϵ/r. Therefore $\sigma_1 + \sigma_2 + \dots + \sigma_r$ is less than $\epsilon + s_1 + s_2 + \dots + s_p$, or than $\epsilon + S$; and since this holds for every value of r, the series $\sigma_1 + \sigma_2 + \dots$ is convergent and its limiting sum is $\leqq S$, since ϵ is arbitrarily small. Also the integer q may be so chosen that the r numbers

$$s_1 - \sum_{n=1}^{n=q} a_{1,n},\ s_2 - \sum_{n=1}^{n=q} a_{2,n},\ \dots\ s_r - \sum_{n=1}^{n=q} a_{r,n}$$

are all less than ϵ/r.

Therefore the limiting sum of the series $\sigma_1 + \sigma_2 + \dots$ is greater than $s_1 + s_2 + \dots + s_r - \epsilon$; and as this holds for each value of r, the limiting sum is $\geqq S - \epsilon$. Since ϵ is arbitrarily small, the limiting sum of the series $\sigma_1 + \sigma_2 + \dots$ is $\geqq S$; and it has been shewn to be $\leqq S$; consequently it is equal to S.

When the positive numbers $a_{r,s}$ are such that each of the series $a_{r,1} + a_{r,2} + \dots$ converges to a number s_r, and so that the series $s_1 + s_2 + \dots$ is convergent, the numbers $a_{r,s}$ are said to be terms of a convergent double series of positive numbers and S is said to be its sum. In accordance with the theorem proved, the limiting sum of the series is the same whether the summation be taken first with respect to s and then with respect to r, or in the converse order. Thus

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{r,s} = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{r,s} = S.$$

If the numbers $a_{r,s}$ are no longer restricted to be of one sign, then if the numbers $|a_{r,s}|$ are the terms of a convergent double series, the numbers $a_{r,s}$ are said to be the terms of an *absolutely convergent* double series.

If the double series, of which the terms are $a_{r,s}$, is absolutely convergent, then

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{r,s} = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{r,s}.$$

For let $a_{r,s} = \beta_{r,s} - \gamma_{r,s}$, where $\gamma_{r,s} = 0$ when $a_{r,s}$ is positive, and $\beta_{r,s} = 0$ when $a_{r,s}$ is negative. The series may be regarded as the difference of two series of which the terms are the positive numbers $\beta_{r,s}$ and $\gamma_{r,s}$. Since the series of which $\beta_{r,s} + \gamma_{r,s}$ is the general term is convergent, the two series of which $\beta_{r,s}$, $\gamma_{r,s}$ are the general terms are both convergent, and their sums may be taken in either order; it follows that the sum of the series of which $a_{r,s}$ is the general term may be taken in either order without affecting the result of summation.

The theorem $\displaystyle\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{r,s} = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty} a_{r,s}$

is also valid when the numbers $a_{r,s}$ are complex, in case the series of moduli $|a_{r,s}|$ is absolutely convergent. For if $a_{r,s} = \gamma_{r,s} + i\delta_{r,s}$, the series of which $\gamma_{r,s}$, $\delta_{r,s}$ are the general terms are both absolutely convergent, whence the result follows.

The general theorem may also be stated as follows:

If $a_1 + a_2 + a_3 + \dots$ be a convergent series of real or complex numbers, and if each term a_r be expressed as the limiting sum of an absolutely convergent series

$$a_{r,1} + a_{r,2} + a_{r,3} + \dots,$$

then the given series may be replaced by the series

$$\sum_{p=1}^{\infty} a_{p,1} + \sum_{p=1}^{\infty} a_{p,2} + \dots,$$

without altering its limiting sum, provided the series

$$S_1 + S_2 + S_3 + \dots$$

is convergent, where S_r denotes the limiting sum of

$$|a_{r,1}| + |a_{r,2}| + |a_{r,3}| + \dots.$$

An important case of this theorem, of which we shall afterwards make use, is the following:

If $a_0 + a_1 z + a_2 z^2 + \dots$ be a convergent series of which the limiting sum is $F(y, z)$, and if $a_0, a_1, a_2, \dots$ are the limiting sums of the absolutely convergent series

$$\begin{gathered} b_{0,0} + b_{0,1}y + b_{0,2}y^2 + b_{0,3}y^3 + \dots \\ b_{1,0} + b_{1,1}y + b_{1,2}y^2 + b_{1,3}y^3 + \dots \\ b_{2,0} + b_{2,1}y + b_{2,2}y^2 + b_{2,3}y^3 + \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{gathered}$$

then, if the series $A_0 + A_1|z| + A_2|z|^2 + \dots$ is convergent, where A_r denotes the sum of the series $|b_{r,0}| + |b_{r,1}y| + |b_{r,2}y^2| + \dots$, the series

$$(b_{0,0} + b_{1,0}z + b_{2,0}z^2 + \dots) + (b_{0,1} + b_{1,1}z + b_{2,1}z^2 + \dots)y \\ + (b_{0,2} + b_{1,2}z + b_{2,2}z^2 + \dots)y^2 + \dots,$$

which is obtained by substituting for $a_0, a_1, a_2, \dots$ in the given series, and arranging the terms as a series in powers of y, is convergent, and its limiting sum is $F(y, z)$ the same as that of the original series.

The Binomial theorem.

211. A very important case of series in ascending integral powers of a variable is the series

$$1 + mz + \frac{m(m-1)}{2!}z^2 + \frac{m(m-1)(m-2)}{3!}z^3 + \dots.$$

In the particular case in which m is a positive integer, the series is finite, and its sum is $(1+z)^m$, the ordinary proof of this being applicable to a complex value of z.

We shall suppose z to be a complex number, but shall confine ourselves to the case in which m is real. In this case a_n/a_{n+1} is equal to $\frac{n+1}{n-m}$, the limiting value of which is unity; the radius of convergence of the series is therefore unity. The series converges absolutely at any point z interior to the circle of radius

unity, and uniformly in any circle of radius less than unity. Denoting the limiting sum of the series by $f(m)$, and applying the theorem of Art. 209, we find for points within the circle of convergence
$$f(m_1)\times f(m_2)=f(m_1+m_2),$$
and thence $f(m_1)f(m_2)\dots f(m_q)=f(m_1+m_2+\dots+m_q)$.

First suppose m to be a positive fraction p/q in its lowest terms, then putting $m_1=m_2=\dots=m_q=p/q$, we have
$$[f(p/q)]^q=f(p),$$
therefore $f(p/q)$ is a qth root of $f(p)$, that is of $(1+z)^p$. Let $1+r\cos\theta=r_1\cos\phi$, $r\sin\theta=r_1\sin\phi$, then
$$(1+z)^p=r_1^p(\cos p\phi+i\sin p\phi),$$
and the values of the qth roots of this are
$$r_1^{\frac{p}{q}}\left\{\cos\frac{p\phi+2s\pi}{q}+i\sin\frac{p\phi+2s\pi}{q}\right\},$$
where s has the values $0, 1, 2, \dots q-1$. We have
$$r_1=+\sqrt{1+2r\cos\theta+r^2},$$
and we may suppose ϕ to be that value of $\tan^{-1}\dfrac{r\sin\theta}{1+r\cos\theta}$ which is acute (positive or negative); such a value exists, for $\cos\phi$ is positive for all points within the circle of convergence. We see then that $f(p/q)$ is a value of $\sqrt[q]{r_1^p}\left\{\cos\dfrac{p\phi+2s\pi}{q}+i\sin\dfrac{p\phi+2s\pi}{q}\right\}$, and s must always have the same value, since we know that $f(p/q)$ is a continuous function for all points within the circle of convergence.

To find the value of s, put $\phi=0$, then $f(p/q)$ is real, and must therefore be equal to a real value of
$$\sqrt[q]{r_1^p}\left\{\cos\frac{2s\pi}{q}+i\sin\frac{2s\pi}{q}\right\},$$
and therefore $s=0$, or $s=\frac{1}{2}q$ in case q is even; if r is sufficiently small, $f\left(\dfrac{p}{q}\right)$ is certainly positive; hence s cannot be equal to $\frac{1}{2}q$ and must therefore be zero.

We have thus proved that the sum of the series, when m is a positive rational number p/q, is the principal value of $(1+z)^{p/q}$, that is
$$(1+2r\cos\theta+r^2)^{p/2q}\left(\cos\frac{p\phi}{q}+i\sin\frac{p\phi}{q}\right),$$

where the expression $(1+2r\cos\theta+r^2)^{p/2q}$ has its real positive value, and ϕ is the numerically smallest value of $\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}$, where

$$z=r(\cos\theta+i\sin\theta).$$

Next let m be a positive irrational number; we consider it to be defined as the limit of a sequence $m_1, m_2, \ldots m_r, \ldots$ of positive rational numbers. It will then be shewn that $f(m)$ is the limit of the sequence $f(m_1), f(m_2), \ldots f(m_r), \ldots$, or $f(m)=Lf(m_r)$. We have, for any point z in the interior of the circle of convergence,

$$f(m_r)=1+m_rz+\frac{m_r(m_r-1)}{2!}z^2+$$
$$+\frac{m_r(m_r-1)\ldots(m_r-n+2)}{(n-1)!}z^{n-1}+R_n(z),$$

where $|R_n(z)|$ is less than the limiting sum of the convergent series

$$\frac{N(N+1)\ldots(N+n-1)}{n!}|z|^n+\frac{N(N+1)\ldots(N+n)}{(n+1)!}|z|^{n+1}+\ldots,$$

where N is a positive number greater than all the numbers $m_1, m_2, \ldots m_r, \ldots$. For all sufficiently great values of n, we have $|R_n(z)|<\epsilon$, for all the numbers m_r, where ϵ is an arbitrarily chosen positive number. It is clear that the limit of the sum of the finite series

$$1+m_rz+\frac{m_r(m_r-1)}{2!}z^2+\ldots+\frac{m_r(m_r-1)\ldots(m_r-n+2)}{(n-1)!}z^{n-1},$$

as m_r converges to m, is

$$1+mz+\frac{m(m-1)}{2!}z^2+\ldots+\frac{m(m-1)\ldots(m-n+2)}{(n-1)!}z^{n-1};$$

and this is therefore the limit of $f(m_r)-R_n(z)$. In accordance with the definition of an irrational power given in Art. 186, the limit of the principal value of $(1+z)^{m_r}$ is $(1+z)^m$. Since $|R_n(z)|<\epsilon$, for all the numbers $m_1, m_2, \ldots m_r, \ldots$, $L|R_n(z)|$, which must have a definite value, is $\leqq \epsilon$.

It follows that

$$1+mz+\frac{m(m-1)}{2!}z^2+\ldots+\frac{m(m-1)\ldots(m-n+2)}{(n-1)!}z^{n-1}$$

differs from the principal value of $(1+z)^m$ by a number of which the modulus is not greater than ϵ, for all sufficiently large values

To shew that when $m > -1$, the absolute magnitude of a_n, diminishes indefinitely as n increases indefinitely, write s for the positive number $m+1$, and denote the product of a certain fixed number of factors in the expression for $|a_n|$ by k; we have then, if r is the integer next greater than s,

$$|a_n| = k\left(1-\frac{s}{r}\right)\left(1-\frac{s}{r+1}\right)\cdots\left(1-\frac{s}{n}\right)$$

$$< k\left[\left(1+\frac{s}{r}\right)\left(1+\frac{s}{r+1}\right)\cdots\left(1+\frac{s}{n}\right)\right]^{-1}$$

$$< k\left[1+s\left(\frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{n}\right)\right]^{-1}.$$

The sum of the first r terms of $\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\ldots$ is $> \frac{1}{2}$, that of the next $2r$ terms is also $> \frac{1}{2}$, and so on; therefore, for a sufficiently large value of n, the sum of the series exceeds a prescribed multiple of $\frac{1}{2}$; and thus the sum of the series increases indefinitely as n does so. It follows that $|a_n|$ diminishes indefinitely as n is increased indefinitely. When $m = -1$, the terms of the binomial series are alternately 1 and -1, and thus the series does not converge.

of n; therefore the convergence of the Binomial series to the principal value of $(1+z)^m$ has been established for the case of a positive irrational number.

Lastly, let m be a negative number $-m_1$. We have then $f(m)f(m_1)=f(0)=1$. Hence $f(m)=1/f(m_1)$, or $f(m)$ is the reciprocal of the principal value of $(1+z)^{m_1}$, or is the principal value of $(1+z)^m$.

We may state the complete result as follows·

The sum of the series

$$1+\mathrm{m}\mathrm{z}+\frac{\mathrm{m}(\mathrm{m}-1)}{2!}\mathrm{z}^2+\ldots+\frac{\mathrm{m}(\mathrm{m}-1)\ldots(\mathrm{m}-\mathrm{n}+1)}{\mathrm{n}!}\mathrm{z}^n+\ldots$$

for all values of z *of which the modulus is less than unity is the principal value of* $(1+\mathrm{z})^{\mathrm{m}}$, *which is*

$$(1+2\mathrm{r}\cos\theta+\mathrm{r}^2)^{\frac{1}{2}m}(\cos \mathrm{m}\phi+\mathrm{i}\sin \mathrm{m}\phi),$$

when m *is any real number,* r *being the modulus and* θ *the argument of* z, *and* ϕ *being that value of* $\tan^{-1}\dfrac{\mathrm{r}\sin\theta}{1+\mathrm{r}\cos\theta}$ *which lies between* $\pm\frac{1}{2}\pi$

This result was obtained by Cauchy, and will be found in his *Analyse Algébrique.*

212. It now remains for us to consider the case when mod. $z=1$.

Denoting the terms of the series

$$1+m+\frac{m(m-1)}{2!}+\frac{m(m-1)(m-2)}{3!}+\ldots$$

by $a_0, a_1, a_2, \ldots$, we have $a_{n+1}/a_n=(m-n)/(n+1)$; when $n>m$ this ratio is negative, therefore the terms of the series are alternately positive and negative, after a fixed term; the series is, by Art. 194, convergent if the terms diminish in absolute magnitude and become ultimately indefinitely small. This will be the case if $n-m<n+1$, that is, if $m>-1$; thus the series is a semi-convergent one, if $m>-1$, whereas if $m<-1$, it is divergent, since the absolute magnitudes of the terms increase indefinitely. From the proposition in Art. 206, it follows that the series $1+mz+\dfrac{m(m-1)}{2!}z^2+\ldots$ converges when mod. $z=1$, provided $m>-1$, and $z\neq-1$.

When $z = -1$, all the terms of the series are, after a certain term, of the same sign; applying the known test

$$Ln(1 + a_n/a_{n-1}) > 1,$$

the series will be convergent if

$$Ln\,\{1 - (n - m - 1)/n\} > 1, \text{ or if } m > 0.$$

According to the theorem in Art. 207, whenever the series

$$1 + mz + \frac{m(m-1)}{2!}z^2 + \,..$$

converges on the circle of convergence, its sum is the value of

$$(1 + 2r\cos\theta + r^2)^{\frac{1}{2}m}(\cos m\phi + i\sin m\phi)$$

at the point. We may state the complete result as follows:

The series

$$1 + \mathrm{m}\mathrm{z} + \frac{\mathrm{m}(\mathrm{m}-1)}{2!}\mathrm{z}^2 + \dots + \frac{\mathrm{m}(\mathrm{m}-1)\dots(\mathrm{m}-\mathrm{n}+1)}{\mathrm{n}!}\mathrm{z}^n + \dots$$

converges when mod. $\mathrm{z} = 1$, *if* m *is positive, for all values of* z; *also if* m *is between* 0 *and* -1, *for all values of* z *except* $\mathrm{z} = -1$, *in which case the argument of* z *is* π. *The series diverges when* $\mathrm{m} = -1$, *and when* $\mathrm{m} < -1$. *For all values of* z *for which the series converges, its sum is* $(2 + 2\cos\theta)^{\frac{1}{2}\mathrm{m}}(\cos\frac{1}{2}\mathrm{m}\theta + \mathrm{i}\sin\frac{1}{2}\mathrm{m}\theta)$, *where* θ *has a value between* $\pm\pi$.

The Binomial Theorem has been considered generally, for complex values of m, by Abel, in a memoir published in *Crelle's Journal*, Vol. I.

The circular functions of multiple angles.

213. An important application of the Binomial Theorem in its generalized form, is the expansion of $(\cos\theta + i\sin\theta)^m$, of which, by De Moivre's Theorem, the principal value is $\cos m\theta + i\sin m\theta$, if θ lies between $\pm\pi$. Writing $(\cos\theta + i\sin\theta)^m$ in the form $\cos^m\theta\,(1 + i\tan\theta)^m$, we have

$$\cos m\theta + i\sin m\theta = \cos^m\theta\left[\left\{1 - \frac{m(m-1)}{2!}\tan^2\theta + \dots\right\}\right.$$
$$\left. + i\left\{m\tan\theta - \frac{m(m-1)(m-2)}{3!}\tan^3\theta + \dots\right\}\right],$$

provided the series is convergent; this condition will be satisfied if θ lies between the limits $\pm\frac{1}{4}\pi$, whatever be the value of m, and also when $\theta = \pm\frac{1}{4}\pi$, provided $m > -1$.

(1) Suppose m positive, then we have

$$\cos m\theta = \cos^m\theta \left\{1 - \frac{m(m-1)}{2\,!}\tan^2\theta \right.$$
$$\left. + \frac{m(m-1)(m-2)(m-3)}{4\,!}\tan^4\theta - \dots\right\} \quad \dots\dots(1),$$

$$\sin m\theta = \cos^m\theta \left\{m\tan\theta - \frac{m(m-1)(m-2)}{3\,!}\tan^3\theta + \dots\right\} \quad \dots\dots(2),$$

for all values of m, provided θ lies between $\pm\frac{1}{2}\pi$, and they hold for $\theta = \pm\frac{1}{2}\pi$. These results are an extension of those obtained in Art. 51, for the case of m a positive integer, in which case there is no convergence condition.

(2) Suppose m negative, then changing m into $-m$ we have

$$\cos m\theta\cos^m\theta = 1 - \frac{m(m+1)}{2\,!}\tan^2\theta$$
$$+ \frac{m(m+1)(m+2)(m+3)}{4\,!}\tan^4\theta - \dots\dots(3),$$

$$\sin m\theta\cos^m\theta = m\tan\theta - \frac{m(m+1)(m+2)}{3\,!}\tan^3\theta + \dots\dots(4),$$

which hold for all positive values of m, provided θ lies between $\pm\frac{1}{2}\pi$. These results hold for $\theta = \pm\frac{1}{2}\pi$, only if m lies between 1 and 0.

214. The formulae (1) and (2) of the last Article have, in the case when m is a positive integer, been applied in Chapter VII to obtain expressions for $\cos m\phi$, $\sin m\phi$, in series of ascending powers of $\sin\phi$. We proceed now to find similar expressions, when m is not a positive integer.

We have proved that, when m is an even positive integer,

$$\cos m\phi = 1 - \frac{m^2}{2\,!}\sin^2\phi + \frac{m^2(m^2-2^2)}{4\,!}\sin^4\phi$$
$$- \frac{m^2(m^2-2^2)(m^2-4^2)}{6\,!}\sin^6\phi + \dots \quad \dots\dots(5),$$

and that, when m is an odd positive integer,

$$\sin m\phi = m\sin\phi - \frac{m(m^2-1^2)}{3\,!}\sin^3\phi$$
$$+ \frac{m(m^2-1^2)(m^2-3^2)}{5\,!}\sin^5\phi - \dots \quad \dots\dots(6).$$

These series were obtained from the expressions for $\cos m\phi$, $\sin m\phi$, in powers of $\cos\phi$ and $\sin\phi$, by substituting for powers of $\cos\phi$, powers of $1-\sin^2\phi$, expanding each of these by the Binomial Theorem for a positive integral index, and arranging the result in powers of $\sin\phi$. The same series will be obtained when m is any positive integer, not limited as to evenness or oddness, provided $\cos\phi$ is positive, which will be the case if ϕ lies between $\pm\frac{1}{2}\pi$; the powers of $1-\sin^2\phi$ will no longer necessarily be integral, but the Binomial Theorem is still applicable since all the series will be convergent. Since all the series of powers of $\sin^2\phi$ are absolutely convergent, and since the original expression for $\cos m\phi$, $\sin m\phi$ each contains only a finite number of terms, by Art. 210, we may arrange the result of the expansions in a series of powers of $\sin^2\phi$. Thus we see that if m is any positive integer, each of the series (5), (6) holds, provided ϕ lies between $\pm\frac{1}{2}\pi$; the first series does not consist of a finite number of terms unless m be even, and the second not unless m be odd.

Let $f(m)$ denote the limiting sum of the series

$$1+m(i\sin\phi)+\frac{m^2}{2!}(i\sin\phi)^2\phi+\frac{m(m^2-1^2)}{3!}(i\sin\phi)^3+\dots,$$

where the series on the right-hand side is obtained by adding the series (5) to the series (6) multiplied by i. When m is a positive integer, we have $f(m)=\cos m\phi+i\sin m\phi$, if ϕ lies between $\pm\frac{1}{2}\pi$. Now when m_1 and m_2 are positive integers, we have

$$\begin{aligned} f(m_1)\times f(m_2) &= (\cos m_1\phi+i\sin m_1\phi)(\cos m_2\phi+i\sin m_2\phi) \\ &= \cos(m_1+m_2)\phi+i\sin(m_1+m_2)\phi \\ &= f(m_1+m_2). \end{aligned}$$

The product of the two series $f(m_1)$, $f(m_2)$ will be of the same form, whatever m_1, m_2 may be; thus, employing the theorem of Art. 209, we conclude that the equation

$$f(m_1)\times f(m_2)=f(m_1+m_2)$$

holds for all values of m_1 and m_2, since the series are absolutely convergent. We have consequently

$$f(m_1)f(m_2)\dots f(m_q)=f(m_1+m_2+\dots+m_q);$$

let $m_1=m_2\dots=m_q=p/q$, where p and q are positive integers, we get then

$$\{f(p/q)\}^q=f(p),$$

hence $f(p/q)$ is a value of $\{f(p)\}^{\frac{1}{q}}$, and is therefore of the form

$$\cos\frac{p\phi+2s\pi}{q}+i\sin\frac{p\phi+2s\pi}{q},$$

where s is some integer. Now when $\phi=0$, we have $f(p/q)=1$ hence since the sum $f(p/q)$ varies continuously as ϕ increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, we must have $s=0$, if ϕ lies between these limits; hence in that case

$$f(p/q)=\cos\frac{p\phi}{q}+i\sin\frac{p\phi}{q}.$$

Next let m be a positive irrational number defined as the limit of a sequence of irrational numbers $m_1, m_2, \dots m_s, \dots$. We have then

$$\begin{aligned}f(m_s)=1+m_s(i\sin\phi)+\frac{m_s^2}{2!}(i\sin\phi)^2+\dots\\ +\frac{m_s(m_s^2-1^2)\dots(m_s^2-\overline{2r-3}|^2)}{(2r-1)!}(i\sin\phi)^{2r-1}\\ +\frac{m_s^2(m_s^2-2^2)\dots(m_s^2-\overline{2r-2}|^2)}{(2r)!}(i\sin\phi)^{2r}+R,\end{aligned}$$

where $|R|$ is less than the modulus of the limiting sum of the convergent series

$$\frac{N(N^2+1^2)\dots(N^2+\overline{2r-1}|^2)}{(2r+1)!}|\sin\phi|^{2r+1}\\ +\frac{N(N^2+2^2)\dots(N^2+\overline{2r}|^2)}{(2r+2)!}|\sin\phi|^{2r+2}+\dots,$$

N denoting a positive number which is greater than all the numbers $m_1, m_2, \dots$. For each fixed value of ϕ, r may be so chosen that $|R|<\epsilon$, where ϵ is an arbitrarily chosen positive number, for all the values $m_1, m_2, \dots$ of m_s.

The limit of $f(m_s)$ or $\cos m_s\phi+i\sin m_s\phi$, as s is indefinitely increased, is $\cos m\phi+i\sin m\phi$. It then follows that

$$\begin{aligned}1+m(i\sin\phi)+\frac{m^2}{2!}(i\sin\phi)^2+\dots\\ +\frac{m(m^2-1^2)\dots(m^2-\overline{2r-3}|^2)}{(2r-1)!}(i\sin\phi)^{2r-1}\\ +\frac{m^2(m^2-2^2)\dots(m^2-\overline{2r-2}|^2)}{(2r)!}(i\sin\phi)^{2r}\end{aligned}$$

differs from $\cos m\phi+i\sin m\phi$ by a number of which the modulus does not exceed ϵ. Since ϵ is arbitrary, it has thus been shewn that for each value of ϕ between $\pm\frac{1}{2}\pi$, the infinite series converges to $\cos m\phi + i\sin m\phi$.

Lastly let m be a negative rational or irrational number $-m_1$. Since $f(m)\times f(m_1) = f(0) = 1$, we have

$$f(m) = 1/(\cos m_1\phi + i\sin m_1\phi) = \cos m\phi + i\sin m\phi.$$

We have shewn thus that the two series

$$\cos m\phi = 1 - \frac{m^2}{2!}\sin^2\phi + \frac{m^2(m^2-2^2)}{4!}\sin^4\phi - \dots\dots(5),$$

$$\sin m\phi = m\sin\phi - \frac{m(m^2-1^2)}{3!}\sin^3\phi + \frac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\phi - \dots\dots\dots\dots(6),$$

hold for all values of ϕ lying between $\pm\frac{1}{2}\pi$, whatever real number m may be.

The series (5), (6) converge absolutely when $\phi = \pm\frac{1}{2}\pi$. For, denoting by a_r the absolute value of the general term of the first series, we have

$$\frac{a_r}{a_{r+1}} = \frac{(2r+1)(2r+2)}{(2r)^2 - m^2} = \left(1 + \frac{3}{2r} + \frac{1}{2r^2}\right)\left(1 - \frac{m^2}{4r^2}\right)^{-1};$$

therefore
$$L\,r\left(\frac{a_r}{a_{r+1}} - 1\right) = \frac{3}{2},$$

and thus in accordance with a known test, the series is convergent. The series (6) may in a similar manner be shewn to converge. In accordance with Abel's theorem in Art. 207, the series (5) and (6) converge to the values $\cos\frac{1}{2}m\pi$, $\pm\sin\frac{1}{2}m\pi$, when $\phi = \pm\frac{1}{2}\pi$.

A similar proof will shew that the two series

$$\cos m\phi/\cos\phi = 1 - \frac{m^2-1^2}{2!}\sin^2\phi + \frac{(m^2-1^2)(m^2-3^2)}{4!}\sin^4\phi - \dots\dots\dots\dots(7),$$

$$\sin m\phi/\cos\phi = m\sin\phi - \frac{m(m^2-2^2)}{3!}\sin^3\phi + \frac{m(m^2-2^2)(m^2-4^2)}{5!}\sin^5\phi - \dots\dots\dots\dots(8),$$

hold for all real values of m, provided ϕ lies between $\pm\frac{1}{2}\pi$.

The series (7), (8) are not valid when $\phi = \pm\frac{1}{2}\pi$.

The series (7) terminates only when m is an odd integer, and (8) only when m is an even integer.

215. If we take the series for $\cos m\phi + i \sin m\phi$, from (5) and (6), and put $z = i \sin \phi$, we have, since $(\cos\phi + i\sin\phi)^m = (\sqrt{1+z^2}+z)^m$, the expansion

$$(\sqrt{1+z^2}+z)^m = 1 + mz + \frac{m^2}{2\,!}z^2 + \frac{m(m^2-1^2)}{3\,!}z^3 + \frac{m^2(m^2-2^2)}{4\,!}z^4 + \ldots$$
$$+ \frac{m(m^2-1^2)\ldots(m^2-\overline{2s-3}|^2)}{(2s-1)\,!}z^{2s-1}$$
$$+ \frac{m^2(m^2-2^2)\ldots(m^2-\overline{2s-2}|^2)}{(2s)\,!}z^{2s} + \ldots.$$

In a similar manner we have from (7) and (8)

$$(\sqrt{1+z^2}+z)^m/\sqrt{1+z^2} = 1 + mz + \frac{m^2-1^2}{2\,!}z^2 + \frac{m(m^2-2^2)}{3\,!}z^3 + \ldots\,.$$
$$+ \frac{m(m^2-2^2)\ldots(m^2-\overline{2s-2}|^2)}{(2s-1)!}z^{2s-1}$$
$$+ \frac{(m^2-1^2)(m^2-3^2)\ldots(m^2-\overline{2s-1}|^2)}{(2s)\,!}z^{2s} + \ldots.$$

It can be shewn that these expansions hold for all real values of m, provided the modulus of z is less than unity. By some writers, these expansions are investigated directly, and then the series (5), (6), (7), (8) are deduced. It is however not easy to investigate these series by elementary methods, except when the modulus of $z/\sqrt{1+z^2}$ is less than unity; we should, with that restriction, obtain the series for $\cos m\phi$, $\sin m\phi$, only when ϕ lies between $\pm\frac{1}{4}\pi$, which is the same restriction that applies to the series (1) and (2). However, by employing the principle of continuity, it is seen that the above expansions are valid in the region $|z| < 1$ of convergence of the series.

216. If in the series (5) and (6), we change ϕ into $\frac{1}{2}\pi - \phi$, we obtain the following series which hold for values of ϕ between 0 and π,

$$\cos m\left(\frac{\pi}{2}-\phi\right) = 1 - \frac{m^2}{2\,!}\cos^2\phi + \frac{m^2(m^2-2^2)}{4\,!}\cos^4\phi - \ldots \quad (9),$$

$$\sin m\left(\frac{\pi}{2}-\phi\right) = m\cos\phi - \frac{m(m^2-1^2)}{3\,!}\cos^3\phi + \ldots \quad \ldots(10).$$

We can now find series which express $\cos m\phi$, $\sin m\phi$, when ϕ

has any value[1]. If $\phi = r\pi + \phi_0$, where ϕ_0 lies between $\pm\frac{1}{2}\pi$, and r is an integer, we have

$$\cos m\phi = \cos mr\pi \cos m\phi_0 - \sin mr\pi \sin m\phi_0;$$

also $\sin\phi = (-1)^r \sin\phi_0$, thus we have, if ϕ lies between $(r \pm \frac{1}{2})\pi$,

$$\cos m\phi = \cos mr\pi\left(1 - \frac{m^2}{2\,!}\sin^2\phi + \ldots\right)$$
$$- \sin(m-1)r\pi\left\{m\sin\phi - \frac{m(m^2-1^2)}{3\,!}\sin^3\phi + \ldots\right\} \ldots(11).$$

Similarly

$$\sin m\phi = \sin mr\pi\left(1 - \frac{m^2}{2\,!}\sin^2\phi + \ldots\right)$$
$$+ \cos(m-1)r\pi\left\{m\sin\phi - \frac{m(m^2-1^2)}{3\,!}\sin^3\phi + \ldots\right\} \ldots(12).$$

From (9) and (10), we obtain in a similar manner

$$\cos m\phi = \cos m(2r+1)\frac{\pi}{2}\left\{1 - \frac{m^2}{2\,!}\cos^2\phi + \ldots\right\}$$
$$+ \cos(m-1)(2r+1)\frac{\pi}{2}\left\{m\cos\phi - \frac{m(m^2-1^2)}{3\,!}\cos^3\phi + \ldots\right\} \;(13),$$

$$\sin m\phi = \sin m(2r+1)\frac{\pi}{2}\left\{1 - \frac{m^2}{2\,!}\cos^2\phi + \ldots\right\}$$
$$+ \sin(m-1)(2r+1)\frac{\pi}{2}\left\{m\cos\phi - \frac{m(m^2-1^2)}{3\,!}\cos^3\phi + \ldots\right\} \;(14),$$

where ϕ lies between $r\pi$ and $(r+1)\pi$.

217. Series of some interest may be derived from (5) and (6), (7) and (8), by giving m particular values[2]. Let $\phi = \frac{1}{2}\pi$, we have then, writing x for m, in (5) and (6),

$$\cos\tfrac{1}{2}\pi x = 1 - \frac{x^2}{2\,!} + \frac{x^2(x^2-2^2)}{4\,!} - \ldots \quad \ldots\ldots(15),$$

$$\sin\tfrac{1}{2}\pi x = x - \frac{x(x^2-1^2)}{3\,!} + \frac{x(x^2-1^2)(x^2-3^2)}{5\,!} - \ldots \;(16).$$

Again letting $m = 2x$, $\phi = \frac{1}{6}\pi$, in (5) and (8), we have

$$\cos\tfrac{1}{3}\pi x = 1 - \frac{x^2}{2\,!} + \frac{x^2(x^2-1^2)}{4\,!} - \frac{x^2(x^2-1^2)(x^2-2^2)}{6\,!} + \ldots \quad (17),$$

$$\sin\tfrac{1}{3}\pi x = \tfrac{1}{2}\sqrt{3}\left\{x - \frac{x(x^2-1^2)}{3\,!} + \frac{x(x^2-1^2)(x^2-2^2)}{5\,!} - \ldots\right\} \;(18).$$

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[1] The formulae (11), (12), (13), (14) were given by D. F. Gregory in the *Cambridge Mathematical Journal*, Vol. IV.

[2] The series in this Article were obtained by Shellbach, see *Crelle's Journal*, Vol. XLVIII.; they have also been discussed by Glaisher in the *Messenger of Mathematics*, Vols. II. and VII. Series equivalent to (15) and (16) are given by M. David in the *Bulletin de la Soc. Math. de France*, Vol. XI.

Various series may be found for powers of π, by expanding $\cos\frac{1}{2}\pi x$, $\sin\frac{1}{2}\pi x, \ldots$ in powers of x, and equating the coefficients of the powers of x to those picked out from the above series; for example from (16) we have, by equating the coefficients of x^3,

$$\frac{\pi^3}{48}=\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{5}\cdot\frac{1.3}{2.4}\left(1+\frac{1}{3^2}\right)+\frac{1}{7}\cdot\frac{1.3.5}{2.4.6}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)+\ldots.$$

Expansion of the circular measure of an angle in powers of its sine.

218. If in the expansions (5) and (6), for $\cos m\phi$, $\sin m\phi$, in powers of $\sin\phi$, we arrange the series as series of ascending powers of m, as we are, by Art. 210, entitled to do, since the series

$$1+\frac{m^2}{2\,!}\sin^2\phi+\frac{m^2(m^2+2^2)}{4\,!}\sin^4\phi+\ldots,$$

$$m\sin\phi+\frac{m(m^2+1^2)}{3\,!}\sin^3\phi+\ldots$$

are convergent, we may equate the coefficients of the various powers of m, to the corresponding coefficients in the expansions of $\cos m\phi$, $\sin m\phi$, in powers of ϕ; we thus obtain from (6)

$$\phi=\sin\phi+\frac{1}{2}\frac{\sin^3\phi}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{\sin^5\phi}{5}+\ldots$$
$$+\frac{1\,.\,3\,.\,5\ldots(2r-1)}{2\,.\,4\,.\,6\ldots2r}\frac{\sin^{2r+1}\phi}{2r+1}+\ldots\;\ldots\ldots(19),$$

and from (5)

$$\phi^2=\sin^2\phi+\frac{2}{3}\frac{\sin^4\phi}{2}+\frac{2\,.\,4}{3\,.\,5}\frac{\sin^6\phi}{3}+\ldots$$
$$+\frac{2\,.\,4\ldots(2r-2)}{3\,.\,5\ldots(2r-1)}\frac{\sin^{2r}\phi}{r}+\ldots\ldots\ldots\ldots\ldots(20),$$

these hold for values of ϕ between $\pm\frac{1}{2}\pi$, or when $\phi=\pm\frac{1}{2}\pi$. We may also write them

$$\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5}+\ldots\quad\ldots\ldots\ldots(19),$$

$$(\sin^{-1}x)^2=x^2+\frac{2}{3}\cdot\frac{x^4}{2}+\frac{2\,.\,4}{3\,.\,5}\frac{x^6}{3}+\ldots\quad\ldots\ldots\ldots(20),$$

where $\sin^{-1}x$, in either equation, is the positive or negative acute angle whose sine is equal to x.

The series (19) was discovered by Newton; the method of proof is that of Cauchy.

219. By changing x into $x+h$ in the series (20), and equating the coefficients of h on both sides of the equation, which process is equivalent to a differentiation with respect to x, and may be justified by employing the theorems of Arts. 210 and 208, we obtain the series

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \dots \quad \dots\dots\dots(21),$$

or putting $\sin\phi$ for x,

$$\phi/\sin\phi\cos\phi = 1 + \frac{2}{3}\sin^2\phi + \frac{2 \cdot 4}{3 \cdot 5}\sin^4\phi + \dots \quad \dots\dots(22),$$

or writing $2\phi = \theta$,

$$\theta/\sin\theta = 1 + \frac{1}{3}(1-\cos\theta) + \frac{1 \cdot 2}{3 \cdot 5}(1-\cos\theta)^2 + \dots$$

which may be written

$$\theta \operatorname{cosec}\theta = 1 + \frac{1}{3}\operatorname{vers}\theta + \frac{1 \cdot 2}{3 \cdot 5}\operatorname{vers}^2\theta + \dots \quad \dots(23).$$

Again, in (22), put $\tan\phi = y$, and we obtain the series

$$\tan^{-1} y = \frac{y}{1+y^2}\left\{1 + \frac{2}{3}\frac{y^2}{1+y^2} + \frac{2 \cdot 4}{3 \cdot 5}\frac{y^4}{(1+y^2)^2} + \dots\right\}\dots(24).$$

Expression of powers of sines and cosines in sines and cosines of multiple angles.

220. We shall now shew how expressions of the form $\cos^m\theta\sin^n\theta$ may be conveniently expressed in cosines or sines of multiples of θ. We shall in the first instance confine ourselves to the case of positive integral values of m and n. Let $z = \cos\theta + i\sin\theta$, then $z^{-1} = \cos\theta - i\sin\theta$, hence $2\cos\theta = z + z^{-1}$, $2i\sin\theta = z - z^{-1}$, and

$$(2\cos\theta)^m(2i\sin\theta)^n = (z+z^{-1})^m(z-z^{-1})^n;$$

if we expand the expression in z, in powers of z and z^{-1}, we can arrange the result in a series of terms of one of the two forms $k(z^r + z^{-r})$, $k(z^r - z^{-r})$, where k is a multiplier depending on m, n, and r; now $z^r = \cos r\theta + i\sin r\theta$, and $z^{-r} = \cos r\theta - i\sin r\theta$, by De Moivre's Theorem, hence

$$k(z^r + z^{-r}) = 2k\cos r\theta, \quad 2k(z^r - z^{-r}) = 2ik\sin r\theta,$$

thus we have the required expression for $\cos^m\theta\sin^n\theta$ in a series of cosines or sines of multiples of θ.

EXAMPLE.

Express $\sin^5\theta\cos^6\theta$ *in series of multiples of* θ.

We have $(2i\sin\theta)^5(2\cos\theta)^6=(z-z^{-1})^5(z+z^{-1})^6=(z^2-z^{-2})^5(z+z^{-1})$ which is equal to $(z^{10}-5z^6+10z^2-10z^{-2}+5z^{-6}-z^{-10})(z+z^{-1})$, or $z^{11}+z^9-5z^7-5z^5+10z^3+10z-10z^{-1}-10z^{-3}+5z^{-5}+5z^{-7}-z^{-9}-z^{-11}$, which is equal to $2i(\sin 11\theta+\sin 9\theta-5\sin 7\theta-5\sin 5\theta+10\sin 3\theta+10\sin\theta)$, therefore $\sin^5\theta\cos^6\theta$ is equal to $\frac{1}{2^{10}}(\sin 11\theta+\sin 9\theta-5\sin 7\theta-5\sin 5\theta$ $+10\sin 3\theta+10\sin\theta)$.

This process may also be arranged thus, writing c for $\cos\theta$, s for $\sin\theta$,

$$\begin{aligned}
(2c)^6&=1+6+15+20+15+\ 6+\ 1,\\
(2is)\,(2c)^6&=1+5+\ 9+\ 5-\ 5-\ 9-\ 5-\ 1,\\
(2is)^2(2c)^6&=1+4+\ 4-\ 4-10-\ 4+\ 4+\ 4+1,\\
(2is)^3(2c)^6&=1+3+\ 0-\ 8-\ 6+\ 6+\ 8-\ 0-3-1,\\
(2is)^4(2c)^6&=1+2-\ 3-\ 8+\ 2+12+\ 2-\ 8-3+2+1,\\
(2is)^5(2c)^6&=1+1-\ 5-\ 5+10+10-10-10+5+5-1-1\,;
\end{aligned}$$

here the powers of z are omitted on the right-hand side, and a figure in any line is obtained by subtracting from the figure just above it the one that precedes the latter.

This very convenient mode of carrying out the numerical calculation is given by De Morgan in his *Double Algebra and Trigonometry.*

221. We can obtain formulae for $(2\cos\theta)^m$ and $(2\sin\theta)^n$, when m is a positive integer, in cosines or sines of multiples of θ, by the method we have employed in the last Article. We have

$$(2\cos\theta)^m=(z+z^{-1})^m=z^m+mz^{m-2}+\frac{m(m-1)}{2!}z^{m-4}+\ldots+z^{-m},$$

hence

$$2^{m-1}\cos^m\theta=\cos m\theta+m\cos(m-2)\theta+\frac{m(m-1)}{2!}\cos(m-4)\theta+\ldots,$$

where the last term is

$$\frac{1}{2}\frac{m}{(\frac{1}{2}m)!\,(\frac{1}{2}m)!}\ \text{or}\ \frac{m!}{(\frac{1}{2}m-1)!\,(\frac{1}{2}m+1)!}\cos\theta,$$

according as m is even or odd.

From

$$(2i\sin\theta)^m=(z-z^{-1})^m=z^m-mz^{m-2}+\frac{m(m-1)}{2!}z^{m-4}-\ldots+(-1)^m z^{-m},$$

we obtain similarly

$$2^{m-1}(-1)^{\frac{m}{2}}\sin^m\theta=\cos m\theta-m\cos(m-2)\theta$$
$$+\frac{m(m-1)}{2!}\cos(m-4)\theta-\ldots+(-1)^{\frac{m}{2}}\frac{m!}{2(\frac{1}{2}m)!\,(\frac{1}{2}m)!}$$

when m is even,

or $2^{m-1}(-1)^{\frac{m-1}{2}}\sin^m\theta = \sin m\theta - m\sin(m-2)\theta$

$$+\frac{m(m-1)}{2!}\sin(m-4)\theta - \ldots + (-1)^{\frac{m-1}{2}}\frac{m!}{(\frac{1}{2}m-1)!\,(\frac{1}{2}m+1)!}\sin\theta$$

when m is odd.

These formulae have already been obtained in Chapter VII.

222. We shall next consider the expansions of $\cos^m\theta$, $\sin^m\theta$ in cosines and sines of multiples of θ, when m is any real number greater than -1. We have from Art. 212,

$$2^m(\pm\cos\tfrac{1}{2}\phi)^m\cos m(\tfrac{1}{2}\phi - k\pi)$$
$$= 1 + m\cos\phi + \frac{m(m-1)}{2!}\cos 2\phi + \frac{m(m-1)(m-2)}{3!}\cos 3\phi + \ldots,$$

$$2^m(\pm\cos\tfrac{1}{2}\phi)^m\sin m(\tfrac{1}{2}\phi - k\pi)$$
$$= m\sin\phi + \frac{m(m-1)}{2!}\sin 2\phi + \frac{m(m-1)(m-2)}{3!}\sin 3\phi + \ldots,$$

where ϕ lies between $(2k-1)\pi$ and $(2k+1)\pi$. Multiplying the first series by $\cos\alpha$, and the second by $\sin\alpha$, and adding, we get

$$2^m(\pm\cos\tfrac{1}{2}\phi)^m\cos(\alpha - \tfrac{1}{2}m\phi + mk\pi) = \cos\alpha + m\cos(\alpha-\phi)$$
$$+\frac{m(m-1)}{2!}\cos(\alpha-2\phi) + \frac{m(m-1)(m-2)}{3!}\cos(\alpha-3\phi) + \ldots,$$

where ϕ lies between $(2k-1)\pi$ and $(2k+1)\pi$. Let $\phi = 2\theta$, then corresponding to the two cases of k even $(=2s)$, and k odd $(=2s+1)$, we have

$$2^m\cos^m\theta\cos(\alpha - m\theta + 2ms\pi)$$
$$= \cos\alpha + m\cos(\alpha - 2\theta) + \frac{m(m-1)}{2!}\cos(\alpha - 4\theta) + \ldots,$$

where θ lies between $2s\pi - \frac{1}{2}\pi$ and $2s\pi + \frac{1}{2}\pi$; and

$$2^m(-\cos\theta)^m\cos(\alpha - m\theta + m\overline{2s+1}\,\pi)$$
$$= \cos\alpha + m\cos(\alpha - 2\theta) + \frac{m(m-1)}{2!}\cos(\alpha - 4\theta) + \ldots,$$

where θ lies between $2s\pi + \frac{1}{2}\pi$ and $2s\pi + \frac{3}{2}\pi$.

In these results, put $\alpha = m\theta$, then we have

$$2^m\cos^m\theta\cos 2ms\pi$$
$$= \cos m\theta + m\cos(m-2)\theta + \frac{m(m-1)}{2!}\cos(m-4)\theta + \ldots \quad (25),$$

where θ lies between $2s\pi - \frac{1}{2}\pi$ and $2s\pi + \frac{1}{2}\pi$; also

$$2^m (-\cos\theta)^m \cos(2s+1)\, m\pi$$
$$= \cos m\theta + m\cos(m-2)\,\theta + \frac{m(m-1)}{2\,!}\cos(m-4)\,\theta + \ldots \quad (26),$$

where θ lies between $2s\pi + \frac{1}{2}\pi$ and $2s\pi + \frac{3}{2}\pi$.

Again, put $\alpha = m\theta + \frac{1}{2}\pi$, then we have

$$2^m \cos^m\theta \sin 2ms\pi$$
$$= \sin m\theta + m\sin(m-2)\,\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\,\theta + \ldots \quad (27),$$

where θ lies between $2s\pi - \frac{1}{2}\pi$ and $2s\pi + \frac{1}{2}\pi$; also

$$2^m (-\cos\theta)^m \sin(2s+1)\, m\pi$$
$$= \sin m\theta + m\sin(m-2)\,\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\,\theta + \ldots \quad (28),$$

where θ lies between $2s\pi + \frac{1}{2}\pi$ and $2s\pi + \frac{3}{2}\pi$.

Next change θ into $\theta - \frac{1}{2}\pi$, and then put $\alpha = m\theta$, we then have

$$2^m \sin^m\theta \cos m(2s + \tfrac{1}{2})\,\pi$$
$$= \cos m\theta - m\cos(m-2)\,\theta + \frac{m(m-1)}{2\,!}\cos(m-4)\,\theta - \ldots \quad (29),$$

where θ lies between $2s\pi$ and $(2s+1)\,\pi$; also

$$2^m (-\sin\theta)^m \cos m(2s + \tfrac{3}{2})\,\pi$$
$$= \cos m\theta - m\cos(m-2)\,\theta + \frac{m(m-1)}{2\,!}\cos(m-4)\,\theta - \ldots \quad (30),$$

where θ lies between $(2s+1)\,\pi$ and $(2s+2)\,\pi$.

Lastly, put $\alpha = m\theta + \frac{1}{2}\pi$, and change θ into $\theta - \frac{1}{2}\pi$, we have then

$$2^m \sin^m\theta \sin m(2s + \tfrac{1}{2})\,\pi$$
$$= \sin m\theta - m\sin(m-2)\,\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\,\theta - \ldots \quad (31),$$

where θ lies between $2s\pi$ and $(2s+1)\,\pi$; also

$$(-2\sin\theta)^m \sin m(2s + \tfrac{3}{2})\,\pi$$
$$= \sin m\theta - m\sin(m-2)\,\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\,\theta - \ldots \quad (32),$$

where θ lies between $(2s+1)\,\pi$ and $(2s+2)\,\pi$.

These series are convergent for all values of θ, if m is positive. If m lies between 0 and -1, the extreme values of θ, $2s\pi \pm \frac{1}{2}\pi$ or $2s\pi$, $(2s+1)\,\pi$ must be excluded, as the series cease to be convergent for those values of θ.

The eight formulae of this Article were given by Abel, in his memoir on the Binomial Theorem, and appear to have been overlooked by subsequent writers.

CHAPTER XV.

THE EXPONENTIAL FUNCTION. LOGARITHMS.

The exponential series.

223. LET us consider the infinite series

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^n}{n!}+\ldots,$$

the limiting sum of which we shall denote by $E(z)$, where z is a complex number $x+iy$. If r is the modulus of z, the series

$$1+r+\frac{r^2}{2!}+\ldots$$

is convergent for all values of r, since the ratio of the $(n+1)$th term to the nth is r/n, which diminishes continually as n increases; consequently the original series is absolutely convergent for all values of z. This series is called the exponential series, and is uniformly convergent in any circle with centre at $z=0$.

224. If we multiply together the two exponential series corresponding to z_1 and z_2, the term of the mth degree in z_1 and z_2 is

$$\frac{z_1{}^m}{m!}+\frac{z_1{}^{m-1}}{(m-1)!}\frac{z_2}{1!}+\frac{z_1{}^{m-2}}{(m-2)!}\frac{z_2{}^2}{2!}+\ldots+\frac{z_2{}^m}{m!}$$

which is equal to $\frac{1}{m!}(z_1+z_2)^m$, by the Binomial Theorem for a positive integral index. We have therefore for the product of the two series, the series

$$1+(z_1+z_2)+\frac{(z_1+z_2)^2}{2!}+\ldots+\frac{(z_1+z_2)^m}{m!}+\ldots$$

which converges to $E(z_1+z_2)$. Now by the theorem in Art. 209, since the exponential series are both absolutely convergent, the product of their sums is equal to the sum of the product series as above formed, therefore

$$E(z_1)\times E(z_2)=E(z_1+z_2)\;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

From this fundamental equation we deduce at once

$$E(z_1) \times E(z_2) \ldots \times E(z_n) = E(z_1 + z_2 + \ldots + z_n)$$

and thence $$\{E(z)\}^n = E(nz) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where n is any positive integer[1].

225. If in the equation (2), we put $z = 1$, we have

$$E(n) = \{E(1)\}^n,$$

where $E(1)$ denotes the limiting sum of the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

It will later on be shewn that the number $E(1)$ is an irrational number $2{\cdot}718281828459\ldots$; it is usually denoted by e. We have therefore when n is a positive integer, $E(n) = e^n$.

Again in (2), let $z = p/q$, where p and q are prime to one another, and let $n = q$, we have then $\{E(p/q)\}^q = E(p)$, hence $E(p/q)$ must be a qth root of $E(p)$ or e^p; since $E(p/q)$ is real and positive, it follows that $E(p/q)$ is the real positive value of $\sqrt[q]{e^p}$, which we call the principal value of $e^{p/q}$.

The exponential series is a particular case of the power series considered in Arts. 203—208. Its radius of convergence is infinite, and consequently the series converges uniformly in any fixed circle with its centre at the point $z = 0$. Moreover, in accordance with the theorem proved in Art. 200, the function $E(z)$ is continuous at any point z. If x be any given irrational positive real number, it can be defined as the limit of a sequence $x_1, x_2, \ldots x_m, \ldots$ of positive rational numbers. In accordance with the definition in Art. 186, the principal value of e^x is the limit of e^{x_m} when the integer m is indefinitely increased; it is known that this limit exists and has a value independent of the particular sequence of rational numbers employed to define the given irrational number x. Since $E(z)$ is a continuous function, it follows that $E(x)$ is the limit of $E(x_m)$ when m is indefinitely increased. Hence since $e^{x_m} = E(x_m)$, for every value of m, it follows that $e^x = E(x)$, when e^x has its principal value.

Next if x be any negative real number, since

$$E(x)E(-x) = E(0) = 1,$$

we have $E(x) = 1/e^{-x} = e^x$, where e^x, e^{-x} have their principal values.

---

[1] This investigation is due to Cauchy, see his *Analyse Algébrique*.

We have thus proved that *for any real number* x, *the sum of the limiting sum of the series* $1 + \mathrm{x} + \frac{\mathrm{x}^2}{2!} + \dots$ *is the principal value of* e^{x}, *where* e *is defined by* $\mathrm{E}(1) = \mathrm{e}$. This is the exponential theorem for a real exponent.

226. We shall now shew that whatever complex number z is, the number $E(z)$, the limiting sum of the exponential series in powers of z, is equal to the limiting value of $(1 + z/m)^m$, where m has positive integral values, when m is indefinitely increased. We have

$$
\begin{aligned}
&(1 + z/m)^m \\
&= 1 + m\frac{z}{m} + \frac{m(m-1)}{2!}\frac{z^2}{m^2} + \dots + \frac{m(m-1)\dots(m-s+1)}{s!}\frac{z^s}{m^s} + \dots \\
&= 1 + z + \left(1 - \frac{1}{m}\right)\frac{z^2}{2!} + \dots + \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\dots\left(1 - \frac{s-1}{m}\right)\frac{z^s}{s!} + \dots.
\end{aligned}
$$

Now if $a, b, c, \dots$ be any positive real numbers, less than unity, we have

$$
\begin{aligned}
(1-a)(1-b) &> 1 - (a+b) \\
(1-a)(1-b)(1-c) &> (1-a-b)(1-c) \\
&> 1 - (a+b+c)
\end{aligned}
$$

....

Hence

$$(1-a)(1-b)(1-c)\dots, < 1, \text{ and } > 1 - (a+b+c+\dots),$$

say

$$= 1 - \theta(a+b+c+\dots),$$

where θ is some number between 0 and 1. Hence we have

$$
\begin{aligned}
\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\dots\left(1 - \frac{s}{m}\right) &= 1 - \theta_s\left(\frac{1}{m} + \frac{2}{m} + \dots + \frac{s}{m}\right) \\
&= 1 - \theta_s . \frac{s(s+1)}{2m},
\end{aligned}
$$

where θ_s is some number between 0 and 1.

We have now

$$(1 + z/m)^m = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^s}{s!} + \dots + \frac{z^m}{m!} + R,$$

where R denotes

$$-\frac{z^2}{2m}\left\{1 + \theta_2 . \frac{z}{1} + \theta_3 . \frac{z^2}{2!} + \dots + \theta_{s+1}\frac{z^s}{s!} + \dots + \theta_{m-1}\frac{z^{m-2}}{(m-2)!}\right\}.$$

The sum of the series in the bracket has a modulus less than the limiting sum of the convergent series $1 + \frac{|z|}{1} + \frac{|z|^2}{2!} + \dots$; and when

m is indefinitely increased, $z^2/2m$ converges to zero. Therefore *the limiting value of* $(1 + z/m)^m$, *when* m *is indefinitely increased, is the function* E(z). The number e is the limiting value of $(1 + 1/m)^m$.

·227. The theorem proved in the last Article gives us the means of finding the value of $E(z)$, where $z = x + iy$, a complex number. We have

$$E(x+iy) = L\left(1+\frac{x+iy}{m}\right)^m;$$

put $1 + x/m = \rho\cos\phi$, $y/m = \rho\sin\phi$, then

$$\left(1+\frac{x+iy}{m}\right)^m = \rho^m(\cos\phi + i\sin\phi)^m = \rho^m(\cos m\phi + i\sin m\phi),$$

by De Moivre's theorem. Also

$$\rho = \sqrt{1+\frac{2x}{m}+\frac{x^2+y^2}{m^2}},$$

and ϕ is the principal value of $\tan^{-1}\dfrac{y}{x+m}$. The limiting value of ρ^m is that of

$$\left(1+\frac{x}{m}\right)^m\left\{1+\frac{y^2}{(x+m)^2}\right\}^{\frac{1}{2}m}$$

or of

$$E(x)\left\{1+\frac{y^2}{m(\sqrt{m}+x/\sqrt{m})^2}\right\}^{\frac{1}{2}m};$$

now suppose that r is a fixed positive number less than $\sqrt{m}+x/\sqrt{m}$, then the limit of

$$\left\{1+\frac{y^2}{m(\sqrt{m}+x/\sqrt{m})^2}\right\}^{\frac{1}{2}m}$$

is between unity and that of

$$\left\{1+\frac{y^2}{mr^2}\right\}^{\frac{1}{2}m},$$

or between 1 and $e^{\frac{1}{2}y^2/r^2}$; now r may be made as large as we please, subject only to the condition $r < \sqrt{m}+x/\sqrt{m}$, hence the limit of

$$\left\{1+\frac{y^2}{(x+m)^2}\right\}^{\frac{1}{2}m}$$

is unity, and therefore that of ρ^m is $E(x)$, which is the principal value of e^x. The limiting value of $m\tan^{-1}\dfrac{y}{x+m}$ is that of $\dfrac{my}{x+m}$, which is y; hence we have $L\left(1+\dfrac{x+iy}{m}\right)^m = e^x(\cos y + i\sin y)$, where e^x has its principal value; thus

$$E(x+iy) = e^x(\cos y + i\sin y).$$

Expansions of the circular functions.

228. If in the last result we put $x=0$, we have

$$E(iy)=\cos y+i\sin y,$$

hence
$$\cos y+i\sin y=1+iy-\frac{y^2}{2!}-i\frac{y^3}{3!}+\dots,$$

or, equating the real and imaginary parts on both sides of the equation, we have

$$\cos y=1-\frac{y^2}{2!}+\frac{y^4}{4!}-\dots+(-1)^s\frac{y^{2s}}{(2s)!}+\dots\dots\dots\dots(3),$$

$$\sin y=y-\frac{y^3}{3!}+\frac{y^5}{5!}\dots+(-1)^s\frac{y^{2s+1}}{(2s+1)!}+\dots\dots\dots(4),$$

the series for $\cos y$ and $\sin y$ expanded in powers of the circular measure y; these series have already been obtained in Art. 99.

We may also write these results in the form

$$\left.\begin{aligned}\cos y&=\tfrac{1}{2}\{E(iy)+E(-iy)\}\\ \sin y&=\frac{1}{2i}\{E(iy)-E(-iy)\}\end{aligned}\right\}\dots\dots\dots\dots\dots(5).$$

The exponential values of the circular functions.

229. If z is a real number, the expression e^z, as defined in Algebra, is multiple-valued except when z is a positive integer. If z is a fraction p/q in its lowest terms, $e^{p/q}$ has q values, the qth roots of e^p; of these values, that one which is real and positive is called the principal value of e^z, and is equal to $E(z)$. When z is an irrational real number, the principal value of e^z is defined, as in Art. 186, as the limit of the sequence formed by the principal values of $e^{z_1}, e^{z_2}, \dots e^{z_r}, \dots$, where $z_1, z_2, \dots z_r, \dots$ is a sequence of rational numbers of which z is the limit. We shall in general understand e^z to have its principal value $E(z)$.

When z is not a real number, no definition of e^z *has as yet been given, and it is so far a meaningless symbol.*

It is convenient however to give by definition a meaning to the symbol e^z or e^{x+iy}. At present we give only a partial definition of the meaning we shall attach to e^z; we define only what may be called its principal value, and shall shortly proceed to a more general definition.

The principal value of the function e^z *we define to be the function* $E(z)$, *or*[1], *what amounts to the same thing, the limit of* $(1 + z/m)^m$, *when* m *is indefinitely increased through positive integral values.*

It should be observed that this definition of the principal value of e^{x+iy} is such that the function satisfies the ordinary indicial law

$$e^{x_1+iy_1} \times e^{x_2+iy_2} = e^{x_1+x_2+i(y_1+y_2)};$$

this follows from the theorem (1) of Art. 224. We shall in general, when we use the symbol e^z, understand it to have its principal value $E(z)$ as just defined.

230. With this understanding as to the meaning of the symbol e^{x+iy}, we have, by Art. 227,

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

and putting $x = 0$, $\quad e^{iy} = \cos y + i \sin y$.

The theorem (5) may now be written

$$\left.\begin{aligned} \cos y &= \tfrac{1}{2}(e^{iy} + e^{-iy}) \\ \sin y &= \frac{1}{2i}(e^{iy} - e^{-iy}) \end{aligned}\right\} \quad \text{...............(6)}.$$

These are called the exponential values of the cosine and sine. The student should bear in mind that these theorems (6) are nothing more than a symbolical mode of writing the equations (3) and (4) which have also been written as in (5).

The only advantage of the symbol e^{iy} over the symbol $E(iy)$ is that the former one reminds us more readily of the law of combination given in Art. 224. The theorem (1) is of the same form as that for the multiplication of real exponentials; we therefore find it convenient to introduce exponentials with imaginary indices, for which the law of combination shall be that expressed by (1).

230⁽¹⁾. The function e^z being defined as above, for any complex value of z, as the limiting sum of the exponential series

$$1 + z + z^2/2! + z^3/3! + \dots,$$

we see that $e^z = 1 + z + \dfrac{z^2}{2!} + \dots + \dfrac{z^s}{s!} + R_s$, where $|R_s|$ is not greater than the sum of the infinite series $\dfrac{|z|^{s+1}}{(s+1)!} + \dfrac{|z|^{s+2}}{(s+2)!} + \dots$. It follows that $|R_s|$ is less than $\frac{|z|^{s+1}}{(s+1)!}\left\{1+|z|+\frac{|z|^2}{2!}+\frac{|z|^3}{3!}+\dots\right\}$, or than $\frac{|z|^{s+1}}{(s+1)!}e^{|z|}$. In case $|z|<1$, we see that

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[1] The latter form of the definition is that introduced by Schlömilch, see *Zeitschrift für Math.* Vol. VI.

$$|R|<\frac{|z|^{s+1}}{(s+1)!}\left\{1+|z|+|z|^2+\dots\right\}$$

or

$$|R|<\frac{|z|^{s+1}}{(s+1)!}\frac{1}{1-|z|}.$$

We have thus shewn that $e^z=1+z+\frac{z^2}{2!}+\dots+\frac{z^s}{s!}(1+u_s)$, where $|u_s|<\frac{|z|}{s+1}e^{|z|}$; and thus $|u_s|$ converges to zero as $|z|$ does so. In particular, by taking $s=1$, we have the theorem $e^z=1+z(1+u_1)$, where $|u_1|<\frac{1}{2}|z|e^{|z|}$; and thus $|u_1|$ converges to zero as $|z|$ does so. We may express this result in the form $\underset{|z|=0}{L}\frac{e^z-1}{z}=1$.

From the last result we have $\underset{h=0}{L}\frac{e^{z+h}-e^z}{h}=e^z$, and thus the function e^z is such that it is equal to its own differential coefficient.

The function e^z may be introduced into Analysis by defining it as that function u which satisfies the conditions $\frac{du}{dz}=u$ for every value of z, and $u=1$ when $z=0$. If it be assumed that there exists a series $a_0+a_1z+a_2z^2+\dots$ which is convergent for every value of z, and such that the derivative series $a_1+2a_2z+3a_3z^2+\dots$ has the same property, both series converge uniformly in a circle of any finite radius. Denoting by u the sum of the first series, that of the second series is, in accordance with a known theorem, $\frac{du}{dz}$. If then $\frac{du}{dz}=u$, we can equate the coefficients of corresponding terms; thus $a_1=a_0$, $2a_2=a_1, \dots na_n=a_{n-1}$; and thence we find $a_n=a_0/n!$. It follows that $u=a_0\left\{1+z+\frac{z^2}{2!}+\dots+\frac{z^n}{n!}+\dots\right\}$; and it is easily seen that this series actually satisfies the assumed conditions of uniform convergence. It follows that the sum of this series satisfies the condition $\frac{du}{dz}=u$. If $u=1$ when $z=0$, we must have $a_0=1$. In this manner we are led to the series

$$1+z+\frac{z^2}{2!}+\dots+\frac{z^n}{n!}+\dots,$$

with the investigation of which we have commenced in the text of this Chapter.

Periodicity of the exponential and circular functions.

231. We have shewn that $E(z) = e^x(\cos y + i \sin y)$; now $\cos y$, $\sin y$ are unaltered if $2k\pi$ be added to y, k being any positive or negative integer, consequently $E(z) = E(z + 2ik\pi)$; or $E(z)$ is a periodic function, of period $2i\pi$. Since $e^z = e^{z+2ki\pi}$, the exponential e^z is periodic, with the imaginary period $2i\pi$; also $e^{iz} = e^{i(z+2k\pi)}$, or e^{iz} as before defined, is a periodic function of z, with a real period 2π.

We have thus seen that each of the two functions e^z, e^{iz} is *singly periodic*, the first having an imaginary period $2i\pi$, and the latter a real period 2π. The student who is acquainted with the elements of Elliptic Functions will know that it is possible to construct functions which have both a real and an imaginary period; such functions are called *doubly periodic*.

232. The circular functions $\cos y$, $\sin y$ were first introduced by means of a geometrical definition, and we have regarded them, in the earlier part of this work, as functions of an angular magnitude measured in circular measure. We can however drop the idea of the angular magnitude, and regard them as functions of a *variable*; a value of the variable of course measures the magnitude in circular measure of an angle by means of which they were defined. The main importance of these functions in Analysis is derived from their property of single periodicity; it has been shewn by Fourier and others that all functions having a real period can, under certain limitations, be represented by means of a series of these circular functions. It would however be beyond the scope of the present work to enter into this most important branch of Analysis.

Analytical definition of the circular functions.

233. It is possible to give purely analytical definitions of the circular functions, and to deduce from these definitions their fundamental analytical properties, so that the calculus of circular functions can be placed upon a basis independent of all geometrical considerations; these definitions will include the circular functions of a complex number.

We can define the cosine and sine of z by means of the equations

$$\left.\begin{aligned} \cos z &= \tfrac{1}{2}\{E(iz)+E(-iz)\} \\ \sin z &= \frac{1}{2i}\{E(iz)-E(-iz)\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(7),$$

where $E(z)$ denotes the limiting sum of the series $1+z+\frac{z^2}{2!}+\ldots$. In other words, we define $\cos z$ as the limiting sum of the series $1-\frac{z^2}{2!}+\frac{z^4}{4!}\ldots$, and $\sin z$ as the limiting sum of the series $z-\frac{z^3}{3!}+\frac{z^5}{5!}\ldots$. We may regard this then as the generalized definition of the cosine and sine functions, and it includes the case of a complex argument, which was not included in the earlier geometrical definitions.

For real values of z, the functions $\cos z$, $\sin z$ are in accordance with the earlier geometrical definitions, because the series which they represent agree with those obtained, in Art. 99, from the geometrical definitions.

By employing the theorem $e^z = 1+z+\frac{z^2}{2!}+\ldots+\frac{z^s}{s!}+R_s$, where $|R_s| < \frac{|z|^{s+1}}{(s+1)!}e^{|z|}$, proved in Art. 230(1), we see by changing z into iz and $-iz$, letting $s=2m+1$ and adding the expressions so obtained, that

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots+(-1)^m\frac{z^{2m}}{(2m)!}+R_m',$$

where $|R_m'| < \frac{|z|^{2m+2}}{(2m+2)!}e^{|z|}$. In particular, we have $\cos z = 1+R_0'$, where $|R_0'| < \frac{|z|^2}{2}e^{|z|}$, and $\cos z = 1-\frac{1}{2}z^2+R_1'$, where

$$|R_1'| < \frac{|z|^4}{4!}e^{|z|}.$$

In case $|z|<1$, we have also $|R_0'| < \frac{|z|^2}{2(1-|z|)}$, and

$$|R_1'| < \frac{|z|^4}{4!(1-|z|)}.$$

Similarly we see that

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots+(-1)^m\frac{z^{2m+1}}{(2m+1)!}+S_m',$$

where $|S_m'| < \frac{|z|^{2m+3}}{(2m+3)!} e^{|z|}$; and in particular $\sin z = z + R_0'$, where $|R_0'| < \frac{|z|^3}{3!} e^{|z|}$, and $\sin z = z - \frac{1}{6} z^3 + R_1'$, where $|S_1'| < \frac{|z|^5}{5!} e^{|z|}$. If $|z| < 1$, we have also $|S_0'| < \frac{|z|^3}{6(1-|z|)}$, $|S_1'| < \frac{|z|^5}{5!(1-|z|)}$.

234. From the definitions given in Art. 233, we can now deduce the fundamental properties of the two functions. We have

$$\cos z + i \sin z = E(iz), \text{ and } \cos z - i \sin z = E(-iz),$$

hence
$$\cos^2 z + \sin^2 z = E(iz)\, E(-iz) = E(0) = 1.$$

Also

$$\begin{aligned}\cos(z_1 + z_2) &= \tfrac{1}{2}\{E(iz_1 + iz_2) + E(-iz_1 - iz_2)\}\\ &= \tfrac{1}{2}\{E(iz_1)\, E(iz_2) + E(-iz_1)\, E(-iz_2)\}\\ &= \tfrac{1}{4}\{E(iz_1) + E(-iz_1)\}\{E(iz_2) + E(-iz_2)\} + \tfrac{1}{4}\{E(iz_1)\\ &\qquad - E(-iz_1)\}\{E(iz_2) - E(-iz_2)\}\end{aligned}$$

or
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

Similarly
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

Thus the addition theorems follow from our definition.

235. Let us now consider the equation $E(z) = 1$. In the first place this equation has no real root except $z = 0$; for it is clear from the definition of $E(z)$ by means of the exponential series that the equation has no positive real root; and it can have no negative real root $-x$ since the positive number x would then also be a root, as is seen from the relation $E(-x)\, E(x) = 1$.

Also the equation $E(z) = 1$ can have no complex root $\alpha + i\beta$, where $|\alpha| > 0$. For, if $\alpha + i\beta$ were a root, so also would be $\alpha - i\beta$, and therefore $E(2\alpha) = E(\alpha + i\beta)\, E(\alpha - i\beta) = 1$, which is impossible, since 2α cannot be a root.

It thus appears that, in case the equation $E(z) = 1$ has roots other than $z = 0$, they must be purely imaginary. In order to shew that the equation has such a root it will be sufficient to shew that the equation $E(i\beta) - E(-i\beta) = 0$, or $\sin \beta = 0$, has a real root other than zero; for, if β be such a root, we have

$$E(2i\beta) = \{E(i\beta)\}^2 = 1,$$

and thus $2i\beta$ would be a root of $E(z) = 1$.

It will be shewn that, if $f(\beta)$ denote the continuous function $\frac{\sin\beta}{\beta}$ represented as the limiting sum of the series

$$1-\frac{\beta^2}{3!}+\frac{\beta^4}{5!}-\frac{\beta^6}{7!}+\ldots,$$

then $f(\beta)$ is positive for all values of β such that $0 \leqq \beta \leqq 3$, and that it is negative when $\beta = 4$. From this it may be concluded that $f(\beta)$ is zero for one value of β between 3 and 4, or for an odd number of such values; and in any case that the numerically smallest positive root of $f(\beta) = 0$ is between 3 and 4, in case the equation has more roots than one.

If β is positive and $< \sqrt{20}$, each term in the series for $f(\beta)$, with the exception of the first, is numerically greater than the next following term. We have therefore $f(\beta) > 1-\frac{\beta^2}{3!}+\frac{\beta^4}{5!}-\frac{\beta^6}{7!}$, for values of β between 0 and some number greater than 3. Denoting $1-\frac{\beta^2}{3!}+\frac{\beta^4}{5!}-\frac{\beta^6}{7!}$ by $\phi(\beta)$, we find that $\phi(3) = 17/560$, which is positive, and $\phi(0) = 1$; also the derived function $\phi'(\beta) \equiv -2\beta\left(\frac{1}{3!}-\frac{2\beta^2}{5!}+\frac{3\beta^4}{7!}\right)$ is negative when β is between 0 and 3, since $\frac{1}{3!}-\frac{2\beta^2}{5!}+\frac{3\beta^4}{7!} > \frac{1}{3!}-\frac{2\beta^2}{5!} > \frac{1}{3!}-\frac{2 . 3^2}{5!} > 0$. Hence $\phi(\beta)$ steadily diminishes from 1 to 17/560 as β increases from 0 to 3; and it follows that $f(\beta)$ cannot vanish for values of β between 0 and 3. We have also

$$f(4) < 1-\frac{4^2}{3!}+\frac{4^4}{5!}-\frac{4^6}{7!}+\frac{4^8}{9!} < 1-\frac{8}{15}-\frac{7}{15}\cdot\frac{256}{189} < 0,$$

and therefore, $f(4)$ being negative, there exists at least one root of $f(\beta)$ between 3 and 4.

Denoting the numerically smallest root of $f(\beta) = 0$ by π, we see that $2\pi i$ is a root of $E(z) = 1$, and that there is no root of this equation with smaller modulus, except $z = 0$.

From the present point of view the number π is defined as the number such that $E(2\pi i) = 1$, and such that no number, different from zero, with smaller modulus exists as a root of $E(z) = 1$. If k be any integer, positive or negative, $E(2k\pi i) = \{E(2\pi i)\}^k = 1$; and hence $2k\pi i$ is also a root of the equation $E(z) = 1$. Also there can exist no root $2p\pi i$, where p lies between k and $k+1$; for in that case we should have

$$E(2p\pi i - 2k\pi i) = E(2p\pi i)\,E(-2k\pi i) = 1;$$

and $2(p-k)\pi i$, which has a smaller modulus than $2\pi i$, would be a root of $E(z)=1$, contrary to the supposition that $2\pi i$ denotes the root with smallest modulus.

It has thus been shewn that all the roots of the equation $E(z)=1$ are of the form $2k\pi i$, where k is a positive or negative integer, and π is a definite number which has been shewn above to lie between 3 and 4.

The number π being thus introduced into the analytical theory, we have, for any value of z,

$$E(z+2\pi i)=E(z)E(2\pi i)=E(z);$$

and therefore the function $E(z)$ is a periodic function, with the imaginary period $2\pi i$.

It follows from the definitions of $\cos z$ and $\sin z$ that they are also periodic, their period being 2π; hence $\cos 2\pi=\cos 0=1$ and $\sin 2\pi=\sin 0=0$. We have of course not verified the identity of π as here defined with the ratio of the circumference of a circle to its diameter. This may however be done by considering the case of a real angle for which the period of the cosine or sine is 2π, according to either definition of the number π.

236. We have also, $E(i\pi)\times E(i\pi)=E(2i\pi)=1$, hence $E(i\pi)$ must equal -1, since it cannot equal $+1$, as $i\pi$ is not a root of $E(z)=1$; also $E(-i\pi)=-1$, hence we have $\cos\pi=-1$, $\sin\pi=0$.

Again $\qquad E(\tfrac{1}{2}i\pi)\times E(\tfrac{1}{2}i\pi)=E(i\pi)=-1,$

and $\qquad E(\tfrac{1}{2}i\pi)\times E(-\tfrac{1}{2}i\pi)=1,$

hence $\qquad E(\tfrac{1}{2}i\pi)=\pm i$ and $E(-\tfrac{1}{2}i\pi)=\mp i,$

therefore $\cos\tfrac{1}{2}\pi=0$, and $\sin\tfrac{1}{2}\pi=\pm 1$; to remove the ambiguity, we remark that if z is real, $\sin z$ is essentially positive between the values $z=0$ and $z=\pi$, as has been shewn in Art. 235; therefore $\sin\tfrac{1}{2}\pi=+1$. Having now obtained the values of the cosine and sine of 0, $\tfrac{1}{2}\pi$, π, 2π, we can, by means of the addition theorems, prove all the ordinary properties of the cosine and sine functions.

The functions $\tan z$, $\cot z$, $\sec z$, $\operatorname{cosec} z$ will now be defined by means of the equations $\tan z=\sin z/\cos z$, $\cot z=\cos z/\sin z$, $\sec z=1/\cos z$, $\operatorname{cosec} z=1/\sin z$, and we can then investigate the properties of these functions in the usual way.

All the properties of the circular functions investigated in Chapters IV, V, and VII, are deduced from the addition formulae and the property of periodicity; it follows that all the properties which are there proved for real arguments hold also for complex arguments.

237. A very important case is that in which the number z is entirely imaginary, and equal to iy; we have then

$$\cos iy = \tfrac{1}{2}(e^y + e^{-y}),\ \sin iy = \frac{i}{2}(e^y - e^{-y}),\ \tan iy = i\frac{e^y - e^{-y}}{e^y + e^y},$$

the expressions $\frac{1}{2}(e^y + e^{-y})$, $\frac{1}{2}(e^y - e^{-y})$, $\dfrac{e^y - e^{-y}}{e^y + e^{-y}}$ are called the hyperbolic cosine, sine and tangent of y, and are written $\cosh y$, $\sinh y$, $\tanh y$ respectively; thus we have

$$\cosh y = \cos iy,\ \sinh y = -i \sin iy,\ \tanh y = -i \tan iy.$$

We shall consider these functions in a special Chapter.

Natural logarithms.

238. If $u = E(z)$ which is a single-valued function of the complex variable z, we may define $z = E^{-1}(u)$ to be the logarithm of u to the base e; this system of logarithms is called the natural system of logarithms. Since $E(z)$ is periodic with respect to z, the inverse function $E^{-1}(z)$ will be multiple-valued to an infinite extent; if $\log u$ is one value of z, the general value $\operatorname{Log} u$ will be given by $\operatorname{Log} u = \log u + 2ik\pi$, since $E(z) = E(z + 2ik\pi)$, where k is any positive or negative integer. In particular, the logarithms of a real positive number x will be $\log x + 2ik\pi$, where $\log x$ denotes its ordinary real logarithm.

239. Let $u_1 = E(z_1)$, $u_2 = E(z_2)$, then since

$$E(z_1) \times E(z_2) = E(z_1 + z_2)$$

the logarithms of the product u_1u_2 are the logarithms of $E(z_1 + z_2)$, that is $z_1 + z_2 + 2ik\pi$, or we have

$$\operatorname{Log} u_1 + \operatorname{Log} u_2 = \operatorname{Log}(u_1u_2) + 2ik\pi.$$

We may suppose the expression $2ik\pi$ included in $\operatorname{Log}(u_1u_2)$, hence we may write this equation

$$\operatorname{Log} u_1 + \operatorname{Log} u_2 = \operatorname{Log}(u_1u_2)$$

in which the particular value of one of the logarithms is determined when those of the other two are given.

Now let $u = \rho(\cos\phi + i\sin\phi)$ where ρ is real, then by the result just proved, we have $\operatorname{Log} u = \operatorname{Log}\rho + \operatorname{Log}(\cos\phi + i\sin\phi)$, and since $E(i\phi) = \cos\phi + i\sin\phi$, $i\phi$ is a value of $\operatorname{Log}(\cos\phi + i\sin\phi)$, and $\log \rho + 2ik\pi$ is the general value of $\operatorname{Log} \rho$, we have therefore $\operatorname{Log} u = \log \rho + i(\phi + 2k\pi)$ for the general value of $\operatorname{Log} u$, where by $\log \rho$ we mean the real value of $\operatorname{Log} \rho$.

If ϕ is restricted to be between the values $-\pi$ and π, we shall call $\log \rho + i\phi$ the principal value of $\operatorname{Log} u$ and shall denote it by $\log u$; we have then the general value $\operatorname{Log} u$ given by

$$\operatorname{Log} u = \log u + 2ik\pi,$$

where $\log u$ is its principal value, and k is any positive or negative integer.

We may write this result

$$\operatorname{Log}(x + iy) = \tfrac{1}{2}\log(x^2 + y^2) + i\left(\tan^{-1}\frac{y}{x} + 2k\pi\right) \ldots\ldots(8).$$

The principal value of the logarithm of a real negative number $-x$ has not been sufficiently defined, since the argument of such a quantity may be either π or $-\pi$; we shall however suppose, for convenience, that for its principal value the argument is π, so that its principal value is $\log x + i\pi$, and the general value of its logarithm is $\log x + (2k + 1) i\pi$.

The general value of the logarithm of a real positive number x is given by $\operatorname{Log} x = \log x + \operatorname{Log} 1 = \log x + 2ik\pi$, the principal value being $\log x$.

The principal value of $\operatorname{Log} i$ is $\frac{1}{2}\pi i$, hence $\operatorname{Log} i = (2k + \frac{1}{2}) i\pi$; the principal value of $\operatorname{Log}(-i)$ is $-\frac{1}{2}\pi i$, hence $\operatorname{Log}(-i) = (2k - \frac{1}{2}) i\pi$.

It is also possible to consider the logarithm of u as a single-valued function of the modulus ρ and the argument ϕ, the latter being supposed to go through all values from $-\infty$ to $+\infty$, not being restricted as above to lying between π and $-\pi$; the logarithm of u is then the single-valued function of ρ and ϕ $\log \rho + i\phi$, and every time ϕ increases by 2π, the logarithm increases by $2i\pi$, and the numerical value of the number u becomes the same as before. The student who is acquainted with the theory of Riemann's surfaces will appreciate the full force of this mode of considering a multiple-valued function as converted into a single-valued one.

The general exponential function.

240. If a be any number, real or complex, *the symbol a^z may be defined to mean* $E(z \operatorname{Log} a)$, where $\operatorname{Log} a$ has any of its infinite number of values; when $\operatorname{Log} a$ has its principal value $\log a$, *we shall call* $E(z \log a)$ *the principal value of* a^z.

Since $E(z\operatorname{Log} a)=1+\frac{z\operatorname{Log} a}{1}+\frac{(z\operatorname{Log} a)^2}{2!}+\dots,$

we have the general exponential theorem

$$a^z=1+\frac{z\operatorname{Log} a}{1}+\frac{z^2(\operatorname{Log} a)^2}{2!}+\dots,$$

and the principal value of a^z is given by

$$a^z=1+\frac{z\log a}{1!}+\frac{z^2(\log a)^2}{2!}+\dots.$$

In the case in which a and z are both real, we have the ordinary form of the exponential theorem

$$a^x=1+\frac{x\log a}{1!}+\frac{x^2(\log a)^2}{2!}+\dots$$

which gives the principal value of a^x.

241. In the particular case $a=e$, we have

$$\operatorname{Log} e=\log e+2ik\pi=1+2ik\pi,$$

and the general meaning of the symbol e^z is $E(z\operatorname{Log} e)$ or $E(z+2ik\pi z)$; the principal value of e^z is $E(z)$, and this is in accordance with the definition of the principal value of e^z given in Art. 229. The general value of e^z is therefore

$$E(z)(\cos 2k\pi z+i\sin 2k\pi z).$$

We shall still continue to use the symbol e^z for its principal value.

242. The general value of a^z, as above defined, is equivalent to $E\{z(\log r+i\theta+2ik\pi)\}$, where $a=r(\cos\theta+i\sin\theta)=\alpha+i\beta$, θ being between $-\pi$ and π; writing $z=x+iy$, we thus have for the general value of $(\alpha+i\beta)^{x+iy}$ the expression

$$E\{x\log r-\theta y-2k\pi y+i(y\log r+x\theta+2\pi kx)\}$$

which is equal to

$$e^{x\log r-\theta y-2k\pi y}\{\cos(y\log r+x\theta+2\pi kx)+i\sin(y\log r+x\theta+2\pi kx)\}.$$

The principal value of $(\alpha+i\beta)^{x+iy}$ is therefore

$$e^{x\log r-\theta y}\{\cos(y\log r+x\theta)+i\sin(y\log r+x\theta)\},$$

where $r=\sqrt{\alpha^2+\beta^2},\ \theta=\tan^{-1}\beta/\alpha.$

The value of $\tan^{-1}\beta/\alpha$, to be taken, is not necessarily its principal value as defined in Art. 38.

If $r=1$, we have for the principal value of $(\cos\theta+i\sin\theta)^{x+iy}$, the function $E\{i\theta(x+iy)\}$ which may be written $\cos(x+iy)\theta+\sin(x+iy)\theta$; this is the extension of De Moivre's theorem to the case of a complex index.

243. In order that the equation $a^{z_1}\times a^{z_2}=a^{z_1+z_2}$ may hold, we must suppose that the values of a^{z_1}, a^{z_2}, $a^{z_1+z_2}$ are those corresponding to the same value of $\operatorname{Log} a$; in that case we have

$$\begin{aligned} a^{z_1}\times a^{z_2} &= E\{z_1(\log a+2ik\pi)\}\times E\{z_2(\log a+2ik\pi)\}\\ &= E\{(z_1+z_2)(\log a+2ik\pi)\}\\ &= a^{z_1+z_2}, \end{aligned}$$

but this will not hold if we take different values of k in the two functions a^{z_1}, a^{z_2}. In particular, the equation $a^{z_1}\times a^{z_2}=a^{z_1+z_2}$ is true of the principal values of the functions.

244. The expression $(a^{z_1})^{z_2}$ is not necessarily a value of $a^{z_1z_2}$, but every value of $a^{z_1z_2}$ is a value of $(a^{z_1})^{z_2}$, for

$$a^{z_1z_2}=E(z_1z_2\operatorname{Log} a)=E\{z_1z_2(\log a+2ik\pi)\}$$

and $$\begin{aligned}(a^{z_1})^{z_2}=E\{z_2\operatorname{Log} a^{z_1}\}&=E\{z_2(z_1\operatorname{Log} a+2ik'\pi)\}\\ &=E\{z_1z_2(\log a+2ik\pi)+2i\,.\,k'\pi z_2\},\end{aligned}$$

hence the values of $a^{z_1z_2}$ are only those of $(a^{z_1})^{z_2}$ in the case $k'=0$. If in every case we take the principal values, then the equation $a^{z_1z_2}=(a^{z_1})^{z_2}$ holds.

If we use the symbols a^z, e^z as equivalent to their principal values $E(z\log' a)$, $E(z)$, as is usually done in practice, then we may, as we have just shewn, perform operations in expressions in which these symbols occur, according to the ordinary rules for indices, as in common Algebra.

EXAMPLE.

If A, B, C, D, ... *be the angular points of a regular polygon of* n *sides, inscribed in a circle of radius* a *and centre* O, *prove that the sum of the angles that* AP, BP, CP, ... *make with* OP *is* $\tan^{-1}\dfrac{a^n\sin n\theta}{a^n\cos n\theta-r^n}$, *where* OP = r, *and the angle* AOP $=\theta$.

We have $$r^n-a^ne^{ni\theta}=\prod_{s=0}^{s=n-1}\left\{r-ae^{i\left(\theta+\frac{2s\pi}{n}\right)}\right\},$$

hence taking logarithms,

$$\begin{aligned}&\log(r^n-a^n\cos n\theta-ia^n\sin n\theta)\\ &\qquad=\sum_{s=0}^{s=n-1}\log\left\{r-a\cos\left(\theta+\frac{2s\pi}{n}\right)-ia\sin\left(\theta+\frac{2s\pi}{n}\right)\right\},\end{aligned}$$

and equating the coefficient of i on both sides of the equation,

$$\tan^{-1}\frac{a^n\sin n\theta}{a^n\cos n\theta-r^n}=\sum_{s=0}^{s=n-1}\tan^{-1}\frac{a\sin\left(\theta+\frac{2s\pi}{n}\right)}{a\cos\left(\theta+\frac{2s\pi}{n}\right)-r},$$

corresponding values of the inverse functions being taken; the expression on the right-hand side is the sum of the angles OP makes with AP, BP, ..., hence this sum is $\tan^{-1}\dfrac{a^n \sin n\theta}{a^n \cos n\theta - r^n}$.

Logarithms to any base.

245. If the principal value of a^z is equal to u, then z is called a logarithm of u to the base a, and may be written $\mathrm{Log}_a u$. Now the principal value of a^z is $E(z \log_e a)$, where $\log_e a$ is the principal logarithm of a to the base e, and if $E(z \log_e a) = u$, we have $z \log_e a = \mathrm{Log}_e u = \log_e u + 2ik\pi$, therefore

$$\mathrm{Log}_a u = \mathrm{Log}_e u/\log_e a = (\log_e u + 2ik\pi)/\log_e a.$$

The principal value of $\mathrm{Log}_a u$ we regard as $\log_e u/\log_e a$, and can denote it by $\log_a u$; hence the general value is

$$\mathrm{Log}_a u = \log_a u + 2ik\pi/\log_e a,$$

a multiple-valued function in which the different values differ by multiples of $2i\pi/\log_e a$. In the particular case $a = e$, the above definition accords with that in Art. 238, giving $\log_e u + 2ik\pi$ for the general value of $\mathrm{Log}_e u$.

Generalized logarithms.

246. We may give the following definition of a logarithm, which is more general than that given in the last Article.

If *any value* of a^z is equal to u, then z is a logarithm of u to the base a, and may be written $[\mathrm{Log}_a u]$ to distinguish it from $\mathrm{Log}_a u$ as used in the last Article. The most general value of a^z is $E(z \,\mathrm{Log}_e a)$, and if this is equal to u, we have

$$z \,\mathrm{Log}_e a = \mathrm{Log}_e u, \text{ or } z(\log_e a + 2ik'\pi) = \log_e u + 2ik\pi,$$

where k and k' are integers. Hence the general value of $[\mathrm{Log}_a u]$ is $\mathrm{Log}_e u/\mathrm{Log}_e a$ or $(\log_e u + 2ik\pi)/(\log_e a + 2ik'\pi)$, which is multiple-valued to an infinite extent, in two ways. The logarithms $\mathrm{Log}_a u$ are therefore included as the particular set of values of $[\mathrm{Log}_a u]$ obtained by putting $k' = 0$. We may call $[\mathrm{Log}_a u]$ the *generalized logarithm* of u to the base a.

247. If $a = e$, we have $[\mathrm{Log}_e u] = (\log_e u + 2ik\pi)/(1 + 2ik'\pi)$ which is the expression for the generalized logarithm of u to the base e. In the more restricted logarithm $\mathrm{Log}_e u$, we have defined z to be a value of $\mathrm{Log}_e u$ when *the principal value* of e^z is equal to u, but in the generalized logarithm $[\mathrm{Log}_e u]$, we consider z to be a value of $[\mathrm{Log}_e u]$ when *any value* of e^z is equal to u.

The generalized value of $[\mathrm{Log}_e 1]$ is $2ik\pi/(1+2ik'\pi)$, and of $[\mathrm{Log}_e(-1)]$ is $(2k+1)i\pi/(1+2ik'\pi)$.

The expression $(\log_e u+2ik\pi)/(1+2ik'\pi)$ may be considered from another point of view. The principal value of $\{E(1+2ik'\pi)\}^{\frac{\log u+2ik\pi}{1+2ik'\pi}}$ is, by the theorem (2), $E(\log u+2ik\pi)$ which is equal to u, hence $(\log u+2ik\pi)/(1+2ik'\pi)$ may be regarded as the logarithm according to the definition in Art. 238, of u to the base $E(1+2ik'\pi)$ which is the principal value not of e but of $e^{1+2ik'\pi}$, so that we have in fact $[\mathrm{Log}\, u]$ equal to the values of $\mathrm{Log}_{E(1+2ik'\pi)} u$, for different values of k'. Thus we may regard the generalized logarithms to the base e, as ordinary logarithms to the base not e but $e^{1+2ik'\pi}$, which though numerically equal to e, has different arguments according to the value of k'.

248. The question was at one time frequently discussed, whether a negative real number can have a real logarithm; thus for example whether $\frac{1}{2}$ can be regarded as the logarithm of $-\sqrt{e}$, the fact being borne in mind that $e^{\frac{1}{2}}$ has the values $\pm\sqrt{e}$. The answer to this question depends on the definition we take of a logarithm; if we take the ordinary definition in Art. 238, that z is a logarithm of u when the principal value of e^z is equal to u, then a negative real number can never have a real logarithm; but if we define a logarithm as in Art. 246, that z is a logarithm of u, when any value of e^z is equal to u, then a negative real number may have a real logarithm. If r be a positive real number, we have

$$[\mathrm{Log}-r]=\frac{\log r+(2k+1)i\pi}{1+2k'i\pi}=\frac{\{\log r+2k'(2k+1)\pi^2\}+i\{(2k+1)\pi-2k'\pi\log r\}}{1+4k'^2\pi^2},$$

and this is real if $\log r=(2k+1)/2k'$. If therefore r be such that $\log r$ is of the form $(2k+1)/2k'$, where k and k' are integers, a value of $[\mathrm{Log}(-r)]$ is real; if $\log r$ is not of this form, we can always find a number r_1 differing as little as we please from r, such that $[\mathrm{Log}(-r_1)]$ has a real value; for a fraction p/q in its lowest terms can always be found which differs by as little as we please from $\log r$; let $\log r'=p/q$, if q be even then $[\mathrm{Log}(-r')]$ has a real value, and $r'=r_1$, but if q be odd, we have $r'=e^{\frac{2sp+1}{2sq}}\times e^{-\frac{1}{2sq}}$, and $e^{-\frac{1}{2sq}}$ can be made as near unity as we please by taking s large enough, or $\log r'$ can be made to differ by as little as we please from $\frac{2sp+1}{2sq}$; therefore a number $\frac{2sp+1}{2sq}=\log r_1$ can be found, which differs as little as we please from $\log r$, such that a value of $[\mathrm{Log}(-r_1)]$ is real. We conclude then that although there is not for every value of r, a value of $[\mathrm{Log}(-r)]$ which is real, we can always find a number r_1 such that r_1-r is as small as we please, and such that a value of $[\mathrm{Log}(-r_1)]$ is real.

The logarithmic series.

249. The principal value of $(1+z)^m$ is $E\{m \log_e(1+z)\}$, but, by Art. 211, the principal value of $(1+z)^m$ is the limiting sum of the series

$$1 + mz + \frac{m(m-1)}{2!}z^2 + \ldots + \frac{m(m-1)\ldots(m-s+1)}{s!}z^s + \ldots$$

provided this series is convergent, which is the case if the modulus of z is less than unity, and also if it is equal to unity, provided $m > 0$; it also converges on the circle of convergence, except at the point $z = -1$, when $0 > m > -1$. Now it has been shewn in Art. 210, that we are entitled to arrange this series in powers of m, without altering its sum, provided the series

$$1 + |m||z| + \frac{|m|(|m|+1)}{2!}|z|^2 + \ldots$$
$$+ \frac{|m|(|m|+1)\ldots(|m|+s-1)}{s!}|z|^s + \ldots$$

is convergent; and this is the case if $|z| < 1$.

Since $E\{m \log_e(1+z)\}$ stands for the sum of the series

$$1 + m\log_e(1+z) + \frac{m^2\{\log_e(1+z)\}^2}{2!} + \ldots,$$

we are, by Art. 208, entitled to equate the coefficients of powers of m in the two series; hence

$$\log_e(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots + (-1)^{s-1}\frac{1}{s}z^s + \ldots \ldots (9).$$

This series, which gives the principal value of $\operatorname{Log}_e(1+z)$, is called the logarithmic series; it has been proved to hold when mod. $z < 1$; also according to Art. 207, the series has still $\log_e(1+z)$ for its sum, when mod. $z = 1$, provided the series is convergent, which is the case unless the argument of z is π.

249$^{(1)}$. Assuming that $|z| < 1$, the series (9) shews that

$$\log_e(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots + (-1)^{s-1}\frac{1}{s}z^s + R_s,$$

where $|R_s|$ cannot exceed the sum of the convergent series $\dfrac{|z|^{s+1}}{s+1} + \dfrac{|z|^{s+2}}{s+2} + \ldots$; and thus $|R_s| < \dfrac{|z|^{s+1}}{s+1}(1 + |z| + |z|^2 + \ldots)$ or $|R_s| < \dfrac{|z|^{s+1}}{s+1}\dfrac{1}{1-|z|}$.

We have thus shewn that when $|z| < 1$,

$$\log_e(1+z) = z - \tfrac{1}{2}z^2 + \dots + \frac{(-1)^{s-1}}{s} z^s (1 + v_s),$$

where $|v_s| < \frac{s}{s+1}\frac{|z|}{1-|z|}$; and thus $|v_s|$ converges to zero as $|z|$ does so.

In particular, taking $s=1$, we have $\log_e(1+z) = z(1+v_1)$, where $|v_1| < \frac{1}{2}\frac{|z|}{1-|z|}$; and thus $|v_1|$ converges to zero as $|z|$ does so. This result may be written in the form $\underset{|z|=0}{L}\frac{\log_e(1+z)-z}{z} = 0$.

If m be any positive real number greater than $|z|$, we have $\left(1+\frac{z}{m}\right)^m = e^{m\log_e(1+z/m)} = e^{z(1+w_1)}$, where $|w_1|$ converges to zero as $|z/m|$ does so. Hence if m have assigned to it the values in any sequence of positive real numbers which increase indefinitely, we see that the limit of $\left(1+\frac{z}{m}\right)^m$ is e^z. This theorem has been proved in Art. 226 only in the particular case in which the numbers m are restricted to be positive integers. This restriction has here been removed.

250. Writing $z = r(\cos\theta + i\sin\theta)$, we have

$$\log_e(1+z) = \log_e(1 + r\cos\theta + ir\sin\theta),$$

and this is equal to

$$\tfrac{1}{2}\log_e(1 + 2r\cos\theta + r^2) + i\tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\},$$

where the inverse tangent has its principal value; we have then the two series

$$\tfrac{1}{2}\log_e(1 + 2r\cos\theta + r^2) = r\cos\theta - \tfrac{1}{2}r^2\cos 2\theta + \tfrac{1}{3}r^3\cos 3\theta - \dots(10),$$

$$\tan^{-1}\{r\sin\theta/(1+r\cos\theta)\} = r\sin\theta - \tfrac{1}{2}r^2\sin 2\theta + \tfrac{1}{3}r^3\sin 3\theta - \dots(11),$$

where $r < 1$, or where $r = 1$ and $\theta \neq \pm\pi$.

If we put $r = 1$, we have

$$\log_e(2\cos\tfrac{1}{2}\theta) = \cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \dots\dots(12),$$

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \dots\dots(13),$$

where θ lies between $\pm\pi$, and cannot equal $\pm\pi$.

If in (11) we change θ into 2θ, we have the theorem

$$\log\cos\theta = -\log 2 + \cos 2\theta - \tfrac{1}{2}\cos 4\theta + \tfrac{1}{3}\cos 6\theta - \dots$$

which holds if θ lies between $\pm\frac{1}{2}\pi$.

Changing θ into $\frac{1}{2}\pi - \theta$, we have

$$\log \sin \theta = -\log 2 - \cos 2\theta - \tfrac{1}{2}\cos 4\theta - \tfrac{1}{3}\cos 6\theta - \dots$$

which holds if θ lies between 0 and π.

The series (13) furnishes an example of discontinuity, owing to the series becoming indefinitely slowly convergent as θ approaches the value π; when $\theta = \pi$, the sum of the series is zero, but when θ is less than π by as small an amount as we please, the sum of the series is $\frac{1}{2}\theta$.

Gregory's series.

251. We have $\log_e(\cos\theta + i\sin\theta) = i\theta$, where θ lies between $\pm\pi$, hence $\log_e \cos\theta + \log_e(1 + i\tan\theta) = i\theta$, or

$$\log_e \cos\theta + i(\tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta \dots) \\ + (\tfrac{1}{2}\tan^2\theta - \tfrac{1}{4}\tan^4\theta + \dots) = i\theta,$$

provided $\tan\theta$ lies between ± 1, which is the case if θ lies between $\pm\frac{1}{4}\pi$, and may equal $\pm\frac{1}{4}\pi$; hence we have, since $\cos\theta$ is positive,

$$\log_e \cos\theta = -\tfrac{1}{2}\tan^2\theta + \tfrac{1}{4}\tan^4\theta - \dots$$

and

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots \quad \dots\dots\dots(14).$$

The latter series is called Gregory's series, and holds if θ lies between $\pm\frac{1}{4}\pi$, both limits being included.

Change θ into $\frac{1}{2}\pi - \theta$, then we have

$$\tfrac{1}{2}\pi - \theta = \cot\theta - \tfrac{1}{3}\cot^3\theta + \tfrac{1}{5}\cot^5\theta - \dots$$

which holds when θ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$. The general expressions for any angle θ are

$$\theta = n\pi + \tan\theta - \tfrac{1}{3}\tan^3\theta + \dots$$

or

$$\theta = (n + \tfrac{1}{2})\pi - \cot\theta + \tfrac{1}{3}\cot^3\theta - \dots,$$

where in the first series n is an integer such that $\theta - n\pi$ lies between $\pm\frac{1}{4}\pi$, and in the second such that $\theta - n\pi$ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

Gregory's theorem may be also written in the form

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots,$$

where x lies between ± 1, and $\tan^{-1} x$ has its principal value.

The series for $\sin^{-1} x$ in powers of x, obtained in Art. 218, may be deduced from Gregory's series. Let $\theta = \sin^{-1} x$, then we have

$$\sin^{-1} x = \frac{x}{(1-x^2)^{\frac{1}{2}}} - \frac{1}{3}\frac{x^3}{(1-x^2)^{\frac{3}{2}}} + \frac{1}{5}\frac{x^5}{(1-x^2)^{\frac{5}{2}}} - \dots\dots \\ + (-1)^r \frac{1}{2r+1}\frac{x^{2r+1}}{(1-x^2)^{\frac{1}{2}(2r+1)}} + \dots\dots$$

if x is less than unity, the series obtained by expanding

$$\frac{1}{2r+1}\frac{x^{2r+1}}{(1-x^2)^{\frac{1}{2}(2r+1)}}$$

in powers of x is absolutely convergent, and the series

$$\frac{|x|}{(1-x^2)^{\frac{1}{2}}}+\frac{1}{3}\frac{|x|^3}{(1-x^2)^{\frac{3}{2}}}+\frac{1}{5}\frac{|x|^5}{(1-x^2)^{\frac{5}{2}}}+\dots\dots$$

is convergent if $|x|<\frac{1}{\sqrt{2}}$; we are therefore entitled to arrange the series in powers of x. We find for the coefficient of $(-1)^r x^{2r+1}$, the expression

$$\frac{1}{2r+1}\left\{1-\frac{2r+1}{2}+\frac{(2r+1)(2r-1)}{2.4}-\dots\dots+(-1)^r\frac{(2r+1)(2r-1)\dots1}{2.4.6\dots2r}\right\};$$

the expression in the brackets is the sum of the first $r+1$ coefficients in the expansion of $(1-y)^{\frac{1}{2}(2r+1)}$ in powers of y, and this is equal to the coefficient of y^r in $(1-y)^{-1}(1-y)^{\frac{1}{2}(2r+1)}$ or $(1-y)^{\frac{1}{2}(2r-1)}$, which is equal to

$$(-1)^r\frac{(2r-1)(2r-3)\dots1}{2.4.6\dots2r}.$$

Hence the coefficient of x^{2r+1} in the expansion of $\sin^{-1}x$ is

$$\frac{1}{2r+1}.\frac{1.3.5\dots(2r-1)}{2.4.6\dots2r};$$

therefore

$$\sin^{-1}x=x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\dots\dots+\frac{1.3.5\dots(2r-1)}{2.4.6\dots2r}\frac{x^{2r+1}}{2r+1}+\dots\dots;$$

this proof only shews that this series holds for values of x between $\pm1/\sqrt{2}$, but by employing the fact that the sum of the series is continuous within its circle of convergence, it can be shewn to hold if x is between ±1.

The quadrature of the circle.

251(a). The famous problem known as that of "squaring the circle," that is, of constructing a square whose area is equal to that of a given circle, is equivalent to that of constructing a straight line of length equal to that of the circumference of a given circle. The construction to be employed in solving the problem is a Euclidean one, involving only the construction of circles and straight lines, in accordance with a Euclidean system of postulates.

The problem may be stated as that of the construction of a straight line whose length is represented by the number π, a given finite straight line being taken to have the length represented by unity. The fact, proved by Lambert, that the number π is irrational, that is not representable in the form p/q, where p and q are integers, is not of itself sufficient to establish the impossibility of constructing the line of length π, because a certain class of straight lines of irrational length is capable of Euclidean construction. A step of fundamental importance in this connection was taken when Liouville[1] established the existence of transcendental numbers, as distinct from algebraical numbers. An *algebraical number* is one which is a root of an algebraical equation of any degree n, with coefficients which are rational numbers; without loss of generality these coefficients may be restricted to be all integers, positive or negative. A *transcendental number* is one which cannot be a root of any algebraical equation with rational (or integral) coefficients. Liouville himself gave examples of transcendental numbers, but the first case in which a number, well known in Analysis, was shewn to be transcendental, was that of the number e, the transcendency of which was established by Hermite. Following Hermite, Lindemann[2] gave a proof that π is a transcendental number. He proved the more general theorem that, if $e^x = y$, the two numbers x and y cannot both be algebraical, except in the case $x = 0$, $y = 1$. Simplified proofs that e and π are transcendental numbers were afterwards given[3] by Hilbert, Hurwitz, and Gordan. A modified form of Gordan's proof will be here given.

The proof that π is a transcendental number is equivalent to the establishment of the impossibility of squaring the circle by means of any geometrical construction in which straight lines and circles are alone employed; or more generally when any algebraical curves may be employed. For any such construction amounts to the exhibition of π as a root of some algebraical equation obtained by combination of the cartesian equations of straight lines and circles or other algebraical curves. The fascination which the problem of "squaring the circle" has exercised for centuries upon many minds is such that Lindemann's proof of the impossibility of the problem under the assumed conditions is a result of great importance in relation to a problem of historic interest.

251^(a)^. To shew that the number e is transcendental, let us assume, if possible, that e satisfies the condition

$$A_0 + A_1 e + A_2 e^2 + \dots + A_n e^n = 0,$$

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[1] *Liouville's Journal*, Vol. XVI. 1851.

[2] *Mathematische Annalen*, Vol. XX. 1882.

[3] *Ibid.* Vol. XLIII. 1893.

where $A_1, A_2, \dots A_n$ are positive or negative integers, and A_0 is a positive integer. In order to shew that this assumption leads to a contradiction, it will be shewn that a number K can be determined such that

$$KA_0 = I_0 + f_0,\ KA_1 e = I_1 + f_1,\ KA_2 e^2 = I_2 + f_2, \dots KA_n e^n = I_n + f_n,$$

where $I_0, I_1, I_2, \dots I_n$ denote positive or negative integers, and $f_1, f_2, \dots f_n$ denote numbers numerically less than unity, and such that $f_1 + f_2 + \dots + f_n$ is numerically less than unity, and where $I_0 + I_1 + \dots + I_n$ is not zero. On multiplying the original equation by K, we have the sum of an integer and a number numerically less than unity equal to zero, which is impossible. To determine the number K, let us consider the expression

$$\phi(x) = \frac{x^{p-1}}{(p-1)!}\{(1-x)(2-x)(3-x)\dots(n-x)\}^p,$$

where p is a prime number greater than n and greater than A_0. We may denote $\phi(x)$, when expanded out in powers of x, by $c_{p-1}x^{p-1} + c_p x^p + \dots + c_{np+p-1}x^{np+p-1}$. Denoting by

$$\phi'(x),\ \phi''(x),\ \dots \phi^{(s)}(x),\ \dots \phi^{np+p-1}(x)$$

the successive derived functions of $\phi(x)$, we see that

$$\phi^p(0),\ \phi^{p+1}(0),\ \dots \phi^{np+p-1}(0)$$

are all multiples of p; but $\phi^{(p-1)}(0)$ is not a multiple of p, since $(n!)^p$ is prime to p. Also, if m denote one of the integers, $1, 2, 3, \dots n$, we see that $\phi(m), \phi'(m), \dots \phi^{p-1}(m)$ all vanish, and $\phi^p(m), \phi^{p+1}(m), \dots \phi^{np+p-1}(m)$ are all integers divisible by p.

Let K_p denote $$\sum_{r=p-1}^{r=np+p-1} r!\,c_r$$

or $$\phi^{(p-1)}(0) + \phi^p(0) + \dots + \phi^{np+p-1}(0);$$

thus K_p is not a multiple of p, since $\phi^{p-1}(0)$ is not divisible by p. It will be shewn that the value of K_p, for a sufficiently large value of the prime number p, is the required number K.

Since A_0 is prime to p, $K_p A_0$ is not a multiple of p.

We have

$$K_p A_m e^m = A_m \sum_{r=p-1}^{r=np+p-1} r!\,c_r e^m$$

$$= A_m \sum_{r=p-1}^{r=np+p-1} c_r \left\{m^r + rm^{r-1} + r(r-1)m^{r-2} + \dots + r! + \frac{m^{r+1}}{r+1} + \frac{m^{r+2}}{(r+1)(r+2)} + \dots\right\}.$$

Now the limiting sum of $\frac{m^{r+1}}{r+1}+\frac{m^{r+2}}{(r+1)(r+2)}+\dots$ is less than that of $m^r\left\{1+m+\frac{m^2}{2!}+\dots\right\}$, or than m^re^m; therefore the limiting sum may be denoted by $m^r\theta_re^m$, where $0<\theta_r<1$. We have then

$$K_pA_me^m=A_m\{\phi(m)+\phi'(m)+\dots+\phi^{np+p-1}(m)\}$$
$$+A_me^m\sum_{r=p-1}^{r=np+p-1}c_r\theta_rm^r;$$

the first term on the right-hand side is a positive or negative integer divisible by p, and the second term is numerically less than $|A_m|e^m\sum_{r=p-1}^{r=np+p-1}|c_r|m^r$, or than

$$|A_m|e^m\frac{m^{p-1}}{(p-1)!}\{(1+m)(2+m)\dots(n+m)\}^p,$$

which cannot exceed

$$|A_m|e^m\frac{n^{p-1}}{(p-1)!}\{(1+n)(2+n)\dots(n+n)\}^p.$$

By choosing p great enough, the number

$$\{n(n+1)(n+2)\dots(n+n)\}^p/(p-1)!$$

may be made as small as we please. Let K be the value of K_p when p is so large that

$$\frac{n^{p-1}}{(p-1)!}\{(1+n)(2+n)\dots(n+n)\}^p\{|A_1|e+|A_2|e^2+\dots+|A_n|e^n\}$$

is less than unity.

We have then $K(A_0+A_1e+A_2e^2+\dots+A_ne^n)$ equal to the sum of an integer which is not divisible by p, an integer which is divisible by p, and a number numerically less than 1; and this is impossible. Since e cannot be a root of an equation

$$A_0+A_1x+\dots+A_nx^n=0,$$

with integral coefficients, it is a transcendental number.

251 (3). If π were a root of an algebraical equation with integral coefficients, $i\pi$ would also be a root of such an equation. Let us assume that $i\pi$ is a root of the equation

$$C(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_s)=0,$$

with integral coefficients; thus $i\pi$ is one of the numbers $\alpha_1, \alpha_2, \dots \alpha_s$.

Since $e^{i\pi}=-1$, we have $(1+e^{\alpha_1})(1+e^{\alpha_2})\dots(1+e^{\alpha_s})=0$; on multiplying out the factors, this is of the form

$$A+e^{\beta_1}+e^{\beta_2}+\dots+e^{\beta_n}=0,$$

where A is a positive integer.

It will be observed that all the symmetrical functions of $C\alpha_1, C\alpha_2, \ldots C\alpha_s$ are integers, therefore all those of $C\beta_1, C\beta_2, \ldots C\beta_n$ are integers. We take

$$\phi(x) = \frac{x^{p-1}}{(p-1)!}\, C^{np+p-1}\{(x-\beta_1)(x-\beta_2)\ldots(x-\beta_n)\}^p,$$

where p is a prime number greater than all the numbers, A, n, C, $C^n\,|\beta_1\beta_2\ldots\beta_n|$.

Denoting $\phi(x)$ by $c_{p-1}x^{p-1} + c_p x^p + \ldots + c_{np+p-1}x^{np+p-1}$, we see that $\phi^p(0)$, $\phi^{p+1}(0), \ldots \phi^{np+p-1}(0)$ are all integral multiples of p, and that $\phi^{p-1}(0)$ is not a multiple of p. Also, if $m \leqq n$, $\phi(\beta_m)$, $\phi'(\beta_m), \ldots \phi^{p-1}(\beta_m)$ are all zero, and $\sum\limits_{m=1}^{m=n}\phi^p(\beta_m)$, $\sum\limits_{m=1}^{m=n}\phi^{p+1}(\beta_m)$, $\sum\limits_{m=1}^{m=n}\phi^{p+np-1}(\beta_m)$ are integers divisible by p.

Let

$$K_p = \sum_{r=p-1}^{r=np+p-1} r!\, c_r = \phi^{(p-1)}(0) + \phi^{(p)}(0) + \ldots + \phi^{np+p-1}(0);$$

thus K_pA is not a multiple of p.

Also

$$K_p e^{\beta_m} = \sum_{r=p-1}^{r=np+p-1} c_r\left\{\beta_m^{\,r} + r\beta_m^{\,r-1} + \ldots + r! + \frac{\beta_m^{\,r+1}}{r+1} + \frac{\beta_m^{\,r+2}}{(r+1)(r+2)} + \ldots\right\} = \phi(\beta_m) + \phi'(\beta_m) + \ldots + \phi^{np+p-1}(\beta_m) + e^{|\beta_m|}\sum_{r=p-1}^{r=np+p-1} c_r\theta_r'\,|\beta_m|^r,$$

where the numbers $|\theta_r'|$ all lie between 0 and 1.

The number $\left|\sum\limits_{r=p-1}^{r=np+p-1} c_r\theta_r'\,|\beta_m|^r\right|$ is numerically less than

$$\sum_{r=p-1}^{r=np+p-1} |c_r|\,|\beta_m|^r$$

or than

$$\frac{\beta^{p-1}}{(p-1)!}\, C^{np+p-1}\{(\beta+|\beta_1|)(\beta+|\beta_2|)\ldots(\beta+|\beta_n|)\}^p,$$

where β is the greatest of the numbers $|\beta_1|, |\beta_2|, \ldots |\beta_n|$.

We now choose p so great that

$$\{e^{|\beta_1|} + e^{|\beta_2|} + \ldots + e^{|\beta_n|}\}\, \frac{\beta^{p-1}}{(p-1)!}\, C^{np+p-1}\{(\beta+|\beta_1|)(\beta+|\beta_2|)\ldots\}^p$$

is less than unity.

Taking the value of K_p for such a prime p as the value of K, we see that $K(A + e^{\beta_1} + e^{\beta_2} + \ldots + e^{\beta_n})$ is expressible as the sum of a multiple of p, an integer not divisible by p, and a number numerically less than unity; it is therefore impossible that it can vanish. It has thus been shewn that π cannot be a root of an algebraical equation with integral coefficients, and it is therefore a transcendental number.

The approximate quadrature of the circle.

252. The problem of the quadrature of the circle, which is equivalent to the determination of π, can be solved to any required degree of approximation, by taking a sufficient number of terms in any one of a large number of series which have been given for π. The simplest series which we can obtain is got by putting $\theta = \frac{1}{4}\pi$, in Gregory's series; we have then

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots$$

which however converges much too slowly to be of any practical use for the calculation of π.

253. If we use the identity $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$, and substitute for $\tan^{-1}\frac{1}{2}$, $\tan^{-1}\frac{1}{3}$ their values from Gregory's series, we have

$$\begin{aligned}\frac{\pi}{4} &= \tfrac{1}{2} - \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{5}(\tfrac{1}{2})^5 - \dots \\ &\quad + \tfrac{1}{3} - \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 - \dots.\end{aligned}$$

This is called Euler's series.

Another series may be obtained from the same identity by substituting for $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ their values from the series

$$\tan^{-1} x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\frac{x^2}{1+x^2} + \frac{2\,.\,4}{3\,.\,5}\left(\frac{x^2}{1+x^2}\right)^2 + \dots\right\}$$

which we have obtained in Art. 219. We have then

$$\begin{aligned}\frac{1}{4}\pi &= \frac{4}{10}\left\{1 + \frac{2}{3}\cdot\frac{2}{10} + \frac{2\,.\,4}{3\,.\,5}\left(\frac{2}{10}\right)^2 + \dots\right\} \\ &\quad + \frac{3}{10}\left\{1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2\,.\,4}{3\,.\,5}\left(\frac{1}{10}\right)^2 + \dots\right\}.\end{aligned}$$

254. Other series obtained in a similar manner have been used by various calculators. Clausen[1] obtained his series from the identity $\frac{1}{4}\pi = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$, using Gregory's series; Machin's series is obtained from

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239};$$

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[1] See a paper "On the calculation of π" by Edgar Frisby in the *Messenger of Math.* Vol. II.

Dase used the identity

$$\tfrac{1}{4}\pi = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\tfrac{1}{8}.$$

A more convenient form of Machin's series was used by Rutherford, who used the identity $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$. Hutton[1] gave the series

$$\pi = 2\cdot4\left\{1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2\,.\,4}{3\,.\,5}\,\frac{1}{10^2} + \dots\right\}$$
$$+ \cdot56\left\{1 + \frac{2}{3}\cdot\frac{2}{100} + \frac{2\,.\,4}{3\,.\,5}\left(\frac{2}{100}\right)^2 + \dots\right\};$$

this is obtained from the expansion of $x\tan^{-1}x$ in powers of $\dfrac{x^2}{1+x^2}$, by putting $x=\frac{1}{2}$ and $x=\frac{1}{3}$, and using Clausen's identity.

Euler has given the series

$$\pi = \frac{28}{10}\left\{1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\,.\,4}{3\,.\,5}\left(\frac{2}{100}\right)^2 + \dots\right\}$$
$$+ \frac{30336}{100000}\left\{1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\,.\,4}{3\,.\,5}\left(\frac{144}{100000}\right)^2 + \dots\right\}$$

which can be deduced from the identity

$$\pi = 20\tan^{-1}\tfrac{1}{7} + 8\tan^{-1}\tfrac{3}{79}.$$

The value of π has been calculated by W. Shanks to 707 decimal places[2].

The continued fraction $\dfrac{1}{1+}\,\dfrac{1^2}{2+}\,\dfrac{3^2}{2+}\,\dfrac{5^2}{2+\dots} = \frac{1}{4}\pi$ was given in 1658 A.D. by Lord Brouncker, the first president of the Royal Society. It is obtained by transforming Gregory's series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ according to the usual rule. Stern[3] has given the continued fraction $\frac{1}{2}\pi = 1 + \dfrac{1}{1+}\,\dfrac{1\,.\,2}{1+}\,\dfrac{2\,.\,3}{1+}\,\dfrac{3\,.\,4}{1+\dots}$.

An interesting account of the history of the subject of the quadrature of the circle will be found in the article "Squaring of the Circle" in the *Encyclopædia Britannica*, 9th edition. See also an article by Glaisher in the *Messenger of Mathematics*, Vol. III. "On the quadrature of the circle A.D. 1580—1630."

Trigonometrical identities.

255. It can be shewn as in Art. 190, Ex. (5), that any identical algebraical relation $f(a, b, c\dots) = 0$, between any number of quantities $a, b, c, \dots$, will lead to two corresponding trigonometrical identities. These will be obtained by giving $a, b, c\ldots$ the complex values

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[1] *Phil. Trans.* 1776.

[2] See *Proc. Royal Soc.* Vols. XXI. XXII.

[3] *Crelle's Journal*, Vol X. See also a note by Sylvester, *Phil. Mag.* 1869.

$$\cos\alpha + i\sin\alpha,\quad \cos\beta + i\sin\beta,\quad \cos\gamma + i\sin\gamma\ldots$$

and reducing the given identity to the form

$$\phi(\alpha, \beta, \gamma\ldots) + i\psi(\alpha, \beta, \gamma\ldots) = 0,$$

whence we obtain the trigonometrical identities

$$\phi(\alpha, \beta, \gamma\ldots) = 0,\quad \psi(\alpha, \beta, \gamma\ldots) = 0,$$

which will involve the sines and cosines of $\alpha, \beta, \gamma\ldots$.

The work of reduction will usually be shortened by using the symbolical forms $e^{i\alpha}, e^{i\beta}\ldots$ instead of $\cos\alpha + i\sin\alpha$, $\cos\beta + i\sin\beta\ldots$.

EXAMPLE.

From the identity $\dfrac{(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{(x-c)(x-a)}{(b-c)(b-a)} + \dfrac{(x-a)(x-b)}{(c-a)(c-b)} = 1$,

deduce the identity

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\sin 2(\theta-a) + \frac{\sin(\theta-\gamma)\sin(\theta-a)}{\sin(\beta-\gamma)\sin(\beta-a)}\sin 2(\theta-\beta)$$
$$+ \frac{\sin(\theta-a)\sin(\theta-\beta)}{\sin(\gamma-a)\sin(\gamma-\beta)}\sin 2(\theta-\gamma) = 0.$$

Let $x = e^{2i\theta}$, $a = e^{2ia}$, $b = e^{2i\beta}$, $c = e^{2i\gamma}$, then we have

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} = \frac{(e^{2i\theta} - e^{2i\beta})(e^{2i\theta} - e^{2i\gamma})}{(e^{2ia} - e^{2i\beta})(e^{2ia} - e^{2i\gamma})} = \frac{(e^{i\overline{\theta-\beta}} - e^{-i\overline{\theta-\beta}})(e^{i\overline{\theta-\gamma}} - e^{-\overline{\theta-\gamma}})}{(e^{i\overline{a-\beta}} - e^{-i\overline{a-\beta}})(e^{i\overline{a-\gamma}} - e^{-i\overline{a-\gamma}})}e^{2i(\theta-a)}$$

or $$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\{\cos 2(\theta-a) + i\sin 2(\theta-a)\};$$

transforming each fraction in this manner and equating the coefficient of i to zero, we obtain the identity to be proved.

The summation of series.

256. When the sum of a finite or an infinite series

$$a_0 + a_1x + a_2x^2 + \ldots$$

is known, we may deduce the sums S_1 and S_2 of the series

$$a_0\cos\alpha + a_1x\cos(\alpha+\theta) + a_2x^2\cos(\alpha+2\theta) + \ldots,$$
$$a_0\sin\alpha + a_1x\sin(\alpha+\theta) + a_2x^2\sin(\alpha+2\theta) + \ldots.$$

For suppose $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots,$

then $$e^{i\alpha}f(xe^{i\theta}) = S_1 + iS_2,$$

and also $$e^{-i\alpha} f(xe^{-i\theta}) = S_1 - iS_2,$$

therefore $$S_1 = \tfrac{1}{2}\{e^{i\alpha} f(xe^{i\theta}) + e^{-i\alpha} f(xe^{-i\theta})\},$$

and $$S_2 = \frac{1}{2i}\{e^{i\alpha} f(xe^{i\theta}) - e^{-i\alpha} f(xe^{-i\theta})\},$$

the values of S_1, S_2 thus obtained, can now be reduced to a real form.

EXAMPLES.

(1) *Sum the series*

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots + x^{n-1}\cos\{\alpha+(n-1)\beta\}.$$

We have $$\frac{1-x^n}{1-x} = 1 + x + x^2 + \ldots + x^{n-1}.$$

Change x into $xe^{i\beta}$ and multiply by $e^{i\alpha}$; we have then

$$e^{i\alpha}\cdot\frac{1-x^n e^{in\beta}}{1-xe^{i\beta}} = e^{i\alpha} + xe^{i(\alpha+\beta)} + x^2 e^{i(\alpha+2\beta)} + \ldots + x^{n-1}e^{i(\alpha+\overline{n-1}\beta)}$$

and similarly we have

$$e^{-i\alpha}\frac{1-x^n e^{-in\beta}}{1-xe^{-i\beta}} = e^{-i\alpha} + xe^{-i(\alpha+\beta)} + x^2 e^{-i(\alpha+2\beta)} + \ldots + x^{n-1}e^{-i(\alpha+\overline{n-1}}$$

therefore the sum of the given series is

$$\frac{1}{2}\left\{e^{i\alpha}\cdot\frac{1-x^n e^{in\beta}}{1-xe^{i\beta}} + e^{-i\alpha}\cdot\frac{1-x^n e^{-in\beta}}{1-xe^{-i\beta}}\right\}$$

or $$\frac{1}{2}\,\frac{e^{i\alpha}(1-x^n e^{in\beta})(1-xe^{-i\beta}) + e^{-i\alpha}(1-x^n e^{-in\beta})(1-xe^{i\beta})}{(1-xe^{i\beta})(1-xe^{-i\beta})},$$

which is equal to

$$\frac{\cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos(\alpha+\overline{n-1}\beta)}{1-2x\cos\beta+x^2}.$$

(2) *Sum the infinite series*

$$\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2\sin(\alpha+2\beta)}{2!} + \ldots + \frac{x^n\sin(\alpha+n\beta)}{n!} + \ldots,$$

We have $$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

put $xe^{i\beta}$ for x, and multiply by $e^{i\alpha}$, we have then

$$e^{xe^{i\beta}+i\alpha} = e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!}e^{i(\alpha+2\beta)} + \ldots + \frac{x^n}{n!}e^{i(\alpha+n\beta)} + \ldots$$

and similarly

$$e^{xe^{-i\beta}-i\alpha} = e^{-i\alpha} + xe^{-i(\alpha+\beta)} + \frac{x^2}{2!}e^{-i(\alpha+2\beta)} + \ldots + \frac{x^n}{n!}e^{-i(\alpha+n\beta)} + \ldots;$$

hence the sum of the given series is

$$\frac{1}{2i}\left\{e^{xe^{i\beta}+i\alpha} - e^{xe^{-i\beta}-i\alpha}\right\}$$

or $$\frac{1}{2i}e^{x\cos\beta}\{e^{i(x\sin\beta+a)}-e^{-i(x\sin\beta+a)}\},$$
which is equal to
$$e^{x\cos\beta}\sin(a+x\sin\beta).$$

257. We shall now give some examples of the application of the exponential expressions for the circular functions to the expansion of expressions in series.

(1) To expand $(1-2x\cos\theta+x^2)^{-1}$ in a series of powers of x, where x is less than unity; we have
$$(1-2x\cos\theta+x^2)^{-1}=(1-xe^{i\theta})^{-1}(1-xe^{-i\theta})^{-1},$$
which expressed in partial fractions is equal to
$$\frac{1}{2i\sin\theta}\left(\frac{e^{i\theta}}{1-xe^{i\theta}}-\frac{e^{-i\theta}}{1-xe^{-i\theta}}\right);$$
expanding each fraction in powers of x, we have
$$\frac{1}{2i\sin\theta}(e^{i\theta}+xe^{2i\theta}+x^2e^{3i\theta}+\dots+x^{n-1}e^{ni\theta}+\dots)$$
$$-\frac{1}{2i\sin\theta}(e^{-i\theta}+xe^{-2i\theta}+\dots+x^{n-1}e^{-ni\theta}+\dots),$$
which is equal to
$$\operatorname{cosec}\theta(\sin\theta+x\sin 2\theta+x^2\sin 3\theta+\dots+x^{n-1}\sin n\theta+\dots).$$

It may be shewn, in a similar manner, that
$$\frac{1-x^2}{1-2x\cos\theta+x^2}=1+2x\cos\theta+2x^2\cos 2\theta+\dots+2x^n\cos n\theta+\dots.$$

(2) To expand $\log_e(1+2x\cos\theta+x^2)$ in powers of x, where x is less than unity; we have
$$\log_e(1+2x\cos\theta+x^2)=\log_e(1+xe^{i\theta})+\log_e(1+xe^{-i\theta});$$
hence expanding each logarithm on the right-hand side, we obtain the formula (9) of Art. 250.

(3) To expand $e^{ax}\sin(bx+c)$ in powers of x, we may write the expression
$$\frac{1}{2i}\{e^{ic}\,.\,e^{(a+ib)x}-e^{-ic}\,.\,e^{(a-ib)x}\}.$$

If we expand $e^{(a+ib)x}$, $e^{(a-ib)x}$ in powers of x, we find the coefficient of x^n to be
$$\frac{1}{2i}\frac{1}{n!}\{e^{ic}(a+ib)^n-e^{-ic}(a-ib)^n\};$$

let $b/a = \tan\alpha$, then the expression becomes

$$\frac{1}{2i}\frac{1}{n\,!}(a^2+b^2)^{\frac{1}{2}n}\left\{e^{i(c+n\alpha)}-e^{i(-c+n\alpha)}\right\}$$

or
$$\frac{1}{n\,!}(a^2+b^2)^{\frac{1}{2}n}\sin(c+n\alpha);$$

this is the coefficient of x^n in the required expansion.

(4) Having given $\sin x = n\sin(x+\alpha)$, to expand x in powers of n, when $n<1$.

We have
$$e^{ix}-e^{-ix}=n\left\{e^{i(x+\alpha)}-e^{-i(x+\alpha)}\right\}$$

or
$$e^{2ix}-1=ne^{-i\alpha}\left\{e^{2i(x+\alpha)}-1\right\},$$

therefore
$$e^{2ix}=\frac{1-ne^{-i\alpha}}{1-ne^{i\alpha}};$$

taking logarithms and expanding the right-hand side, we have

$$2i(x+k\pi)=n(e^{i\alpha}-e^{-i\alpha})+\frac{n^2}{2}(e^{2i\alpha}-e^{-2i\alpha})+\ldots,$$

hence
$$x+k\pi=n\sin\alpha+\tfrac{1}{2}n^2\sin 2\alpha+\tfrac{1}{3}n^3\sin 3\alpha+\ldots,$$

where k is an integer.

If B be the angle of a triangle and be less than A, we can expand the circular measure of B in powers of b/a; since

$$\sin B=\frac{b}{a}\sin(B+C),$$

we have, since in this case $k=0$,

$$B=\frac{b}{a}\sin C+\tfrac{1}{2}\frac{b^2}{a^2}\sin 2C+\tfrac{1}{3}\frac{b^3}{a^3}\sin 3C+\ldots,$$

EXAMPLES ON CHAPTER XV.

1. Prove that the general term in the expansion of $\dfrac{A+Bz}{1-2z\cos\phi+z^2}$ in powers of z is $\dfrac{A\sin(n+1)\phi+B\sin n\phi}{\sin\phi}z^n$, and that the general term in the expansion of $\dfrac{A+Bz}{(1-2z\cos\phi+z^2)^2}$ is

$$\frac{(n+3)\sin(n+1)\phi-(n+1)\sin(n+3)\phi}{4\sin^3\phi}Az^n+\frac{(n+2)\sin n\phi-n\sin(n+2)\phi}{4\sin^3\phi}Bz^n.$$

(*Euler.*)

2. If $\tan x = \dfrac{n \sin a}{1 - n\cos a}$, prove that $x = n\sin a + \frac{1}{2}n^2 \sin 2a + \frac{1}{3}n^3 \sin 3a + \ldots$, n being less than unity.

3. If $\cot y = \cot x + \operatorname{cosec} a \operatorname{cosec} x$, shew that

$$y = \sin x \sin a - \tfrac{1}{2}\sin 2x \sin^2 a + \tfrac{1}{3}\sin 3x \sin^3 a - \ldots.$$

4. If $\tan \frac{1}{2}\theta = \left(\dfrac{1+a}{1-a}\right)^{\frac{1}{2}} \tan \frac{1}{2}\phi$, shew that

$$\theta = \phi + 2\lambda \sin\phi + \frac{2\lambda^2}{2}\sin 2\phi + \frac{2\lambda^3}{3}\sin 3\phi + \ldots,$$

where
$$\lambda = \frac{a}{2} + \left(\frac{a}{2}\right)^3 + 2\left(\frac{a}{2}\right)^5 + 5\left(\frac{a}{2}\right)^7 + \ldots.$$

5. If $\tan\theta = x + \tan a$, prove that

$$\theta = a + x\cos^2 a - \tfrac{1}{2}x^2\cos^2 a \sin 2a - \tfrac{1}{3}x^3 \cos^3 a \cos 3a + \tfrac{1}{4}x^4 \cos^4 a \sin 4a + \ldots.$$

6. If $(1+m)\tan\theta = (1-m)\tan\phi$, when θ and ϕ are positive acute angles, shew that

$$\theta = \phi - m\sin 2\phi + \tfrac{1}{2}m^2 \sin 4\phi - \tfrac{1}{3}m^3 \sin 6\phi + \ldots.$$

7. If $\tan a = \cos 2\omega \tan\lambda$, shew that

$$\lambda - a = \tan^2\omega \sin 2a + \tfrac{1}{2}\tan^4\omega \sin 4a + \tfrac{1}{3}\tan^6\omega \sin 6a + \ldots.$$

8. If $\sin x = n\cos(x+a)$, expand x in ascending powers of n.

9. Shew that the coefficient of x^p in the expansion of $(1 - 2x\cos\theta + x^2)^{-n}$ is

$$2\{a_p \cos p\theta + a_1 a_{p-1}\cos(p-2)\theta + a_2 a_{p-2}\cos(p-4)\theta + \ldots\},$$

where a_m is the coefficient of x^m in the expansion of $(1-x)^{-n}$

10. Prove that $\pi^2 = 18 \sum\limits_{n=0}^{n=\infty} \dfrac{n!\,n!}{(2n+2)!}$.

11. Prove that in any triangle

$$\log c = \log a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \ldots,$$

supposing b to be less than a.

12. If the roots of the equation $ax^2 + bx + c = 0$ be imaginary, shew that the coefficient of x^n in the development of $(ax^2+bx+c)^{-1}$ in powers of x is

$$\frac{a^{\frac{1}{2}n}\sin(n+1)\theta}{c^{\frac{1}{2}n+1}\sin\theta},$$

where θ is given by $b\sec\theta + 2\sqrt{ac} = 0$.

13. If $\rho^2 = \dfrac{(1+n)^4\cos^2\theta + (1-n)^4\sin^2\theta}{(1+n)^2\cos^2\theta + (1-n)^2\sin^2\theta}$, expand $\log_e \rho$ in a series of cosines of even multiples of θ.

14. Expand $\log_e \cos(\theta + \frac{1}{4}\pi)$ in a series of sines and cosines of multiples of θ.

15. Prove that

$$\frac{\pi}{4}=\frac{17}{21}-\frac{713}{81.343}+\ldots+\frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}9^{1-n}+7^{1-2n}\right\}+\ldots.$$

16. Prove that

$$1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\ldots=\frac{\pi(\sqrt{2}+1)}{8}.$$

17. Find all the values of $(\sqrt{-1})^{\sqrt{-1}}$.

18. Prove that $(a+a\sqrt{-1}\tan\phi)^{\log_e(a\sec\phi)-\phi\sqrt{-1}}$ is a real number, and find its value.

19. If $a\cos\theta+b\sin\theta=c$, where $c>\sqrt{a^2+b^2}$, shew that

$$\theta=(4n+1)\frac{\pi}{2}+i\log_e\frac{c+\sqrt{c^2-a^2-b^2}}{\sqrt{a^2+b^2}}-\tan^{-1}\frac{a}{b}.$$

20. From the expression for x^n+1 in factors, deduce that when n is even

$$\tan^{-1}\frac{\sin n\theta}{1+\cos n\theta}$$

$$=\tan^{-1}\frac{\sin 2\theta-2\cos\frac{\pi}{n}\sin\theta}{1+\cos 2\theta-2\cos\frac{\pi}{n}\cos\theta}+\tan^{-1}\frac{\sin 2\theta-2\cos\frac{3\pi}{n}\sin\theta}{1+\cos 2\theta-2\cos\frac{3\pi}{n}\sin\theta}+\ldots.$$

21. From the identity $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{(x-a)(x-b)}$ deduce

$$\cos(\theta+\alpha)\sin(\theta-\beta)-\cos(\theta+\beta)\sin(\theta-\alpha)=\sin(\alpha-\beta)\cos 2\theta,$$
$$\sin(\theta+\alpha)\sin(\theta-\beta)-\sin(\theta+\beta)\sin(\theta-\alpha)=\sin(\alpha-\beta)\sin 2\theta.$$

22. Prove that

$$\frac{\tan^{-1}\alpha}{\alpha}+\frac{\tan^{-1}\beta}{\beta}+\frac{\tan^{-1}\gamma}{\gamma}=\frac{\pi}{2}+\frac{\sqrt{3}}{4}\log\frac{2+\sqrt{3}}{2-\sqrt{3}}=3\left(1-\frac{1}{7}+\frac{1}{13}-\frac{1}{19}+\frac{1}{25}-\ldots\right),$$

where α, β, γ are the three cube roots of unity.

23. Express the logarithms of $c+di$ to the base $a+bi$, in the form $A+Bi$.

24. If $\tan^m(\frac{1}{4}\pi+\frac{1}{2}\psi)=\tan^n(\frac{1}{4}\pi+\frac{1}{2}\phi)$, shew that

$$m\tan^{-1}\frac{\sin\psi}{i}=n\tan^{-1}\frac{\sin\phi}{i}.$$

25. In any triangle, shew that

$$a^n\cos nB+b^n\cos nA=c^n-nabc^{n-2}\cos(A-B)$$
$$+\frac{n(n-3)}{2!}a^2b^2c^{n-4}\cos 2(A-B)-\ldots,$$

n being a positive integer.

26. If $$\log_e\log_e\log_e(\alpha+i\beta)=p+iq,$$

then $$e^{e^p\cos q}\cos(e^p\sin q)=\frac{1}{2}\log_e(\alpha^2+\beta^2),$$

and $$e^{e^p\sin q}\sin(e^p\sin q)=\tan^{-1}\frac{\beta}{\alpha}.$$

27. Shew that the coefficient of x^n in the expansion of $e^x \cos x$ in ascending powers of x is $\dfrac{2^{\frac{1}{2}n}}{n!}\cos\dfrac{n\pi}{4}$.

28. Prove that

$$\frac{1}{(1+e\cos\theta)^2}=\sec^3 2\lambda+\ldots+(-1)^n 2\sec^3 2\lambda\tan\lambda\,(1+n\cos 2\lambda)\cos n\theta+\ldots,$$

where 2λ is the least positive value of $\sin^{-1}e$.

29. Prove that the series

$$\frac{1}{1.3.5\ldots(2m+1)}-\frac{1}{3.5.7\ldots(2m+3)}+\ldots\ ad\ inf.$$

can be expressed in the form $\dfrac{A_m\pi+B_m}{C_m}$, where A_m, B_m, C_m are whole numbers, and

$$A_m=1\ 3.5\ldots(2m-1),\quad C_m=\frac{(2m)!}{2^{m-2}},$$

$$B_m=(2m-1)B_{m-1}-2(m-1)!.$$

30. Prove that

$$\sin^n\theta\cos n\phi=\sin^n\phi\cos n\theta+n\sin^{n-1}\phi\cos(n-1)\theta\sin(\theta-\phi)$$
$$+\frac{n(n-1)}{2!}\sin^{n-2}\phi\cos(n-2)\theta\sin^2(\theta-\phi)+\ldots+\sin^n(\theta-\phi),$$

n being a positive integer.

31. Prove the identity

$$\Sigma\frac{\cos 2a}{\sin\frac{1}{2}(a-\beta)\sin\frac{1}{2}(a-\gamma)\sin\frac{1}{2}(a-\delta)}=8\sin\tfrac{1}{2}(a+\beta+\gamma+\delta).$$

32. Prove that $\quad 1+\dfrac{1}{3}-\dfrac{1}{5}-\dfrac{1}{7}+\ldots=\dfrac{\pi}{2\sqrt{2}}$.

33. Reduce $\tan^{-1}(\cos\theta+i\sin\theta)$ to the form $a+bi$, and hence shew that

$$\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta-\ldots=\pm\frac{\pi}{4},$$

the upper or lower sign being taken, according as $\cos\theta$ is positive or negative.

34. Prove that one value of $\text{Log}_e(1+\cos 2\theta+i\sin 2\theta)$ is $\log_e(2\cos\theta)+i\theta$, when θ lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Deduce Gregory's series.

Prove that one value of $\sin^{-1}(\cos\theta+i\sin\theta)$ is

$$\cos^{-1}\sqrt{\sin\theta}+i\log_e(\sqrt{\sin\theta}+\sqrt{1+\sin\theta}),$$

when θ lies between 0 and $\frac{1}{2}\pi$.

35. Find the sum of the series $\sum\limits_0^\infty A_n e^{(2n+1)x}\sin(2n+1)y$ in which

$$A_n=\frac{2}{2n+1}-\frac{1}{2n-1}-\frac{1}{2n+3}.$$

36. In any triangle, shew that if $a<c$

$$\frac{\cos nA}{b^n}=\frac{1}{c^n}\left\{1+n\frac{a}{c}\cos B+\frac{n(n+1)}{2!}\frac{a^2}{c^2}\cos 2B\right.$$
$$\left.+\frac{n(n+1)(n+2)}{3!}\frac{a^3}{c^3}\cos 3B+\ldots\right\}.$$

37. Prove that

$$(\tan^{-1}x)^2=x^2-\tfrac{1}{2}(1+\tfrac{1}{3})x^4+\tfrac{1}{3}(1+\tfrac{1}{3}+\tfrac{1}{5})x^6-\ldots$$
$$+\frac{(-1)^{n-1}}{n}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)x^{2n}+\ldots$$

where x lies between ± 1.

38. If $$u=\log_e \tan\left(\frac{\pi}{4}+\tfrac{1}{2}x\right)=x+a_3x^3+a_5x^5+\ldots,$$

prove that $$x=u-a_3u^3+a_5u^5-\ldots.$$

39. Rationalize $\tan\left\{ic\log_e\dfrac{a-bi}{a+bi}\right\}$.

40. Prove that

$$\frac{\cos x}{(n-1)!\,(n+1)!}+\frac{\cos 2x}{(n-2)!\,(n+2)!}+\ldots+\frac{\cos nx}{(2n)!}=\frac{2^{n-1}(1+\cos x)^n}{(2n)!}-\frac{1}{2(n!)^2}.$$

41. If n is a positive integer, and

$$S=1+n\cos^2\theta+\ldots+\frac{(n+r-2)!}{(n-1)!\,(r-1)!}\cos^{r-1}\theta\cos(r-1)\theta+\ldots,$$

prove that

$$2S\sin^n\theta=\{1+(-1)^n\}(-1)^{\frac{1}{2}n}\cos n\theta+\{1-(-1)^n\}(-1)^{\frac{1}{2}(n-1)}\sin n\theta.$$

42. Prove that the expansion of tan tan tan ... tan x, (n tangents) is

$$x+2n\frac{x^3}{3!}+4n(5n-1)\frac{x^5}{5!}+\frac{8n}{3}(175n^2-84n+11)\frac{x^7}{7!}+\ldots.$$

43. If $\tan(\tfrac{1}{4}a-\phi)=\tan^3\tfrac{1}{4}a$, then shew that

$$\phi=\frac{1}{1.3}\sin a-\frac{1}{2.3^2}\sin 2a+\frac{1}{3.3^3}\sin 3a-\ldots.$$

44. Shew that, if $\tan\theta<1$,

$$\tan^2\theta-\tfrac{1}{2}\tan^4\theta+\tfrac{1}{3}\tan^6\theta-\ldots=\sin^2\theta+\tfrac{1}{2}\sin^4\theta+\tfrac{1}{3}\sin^6\theta+\ldots.$$

45. Prove that, n being a positive integer,

$$1+\frac{n(n-1)(n-2)}{3!}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}+\ldots$$
$$=\frac{1}{3}\left\{2^n+(-1)^n.2\cos\frac{2n\pi}{3}\right\}.$$

46. Shew that the equations

$$x^2\sin 2a+y^2\sin 2\beta+z^2\sin 2\gamma-2yz\sin(\beta+\gamma)-2zx\sin(\gamma+a)-2xy\sin(a+\beta)=0,$$
$$x^2\cos 2a+y^2\cos 2\beta+z^2\cos 2\gamma-2yz\cos(\beta+\gamma)-2zx\cos(\gamma+a)-2xy\cos(a+\beta)=0$$

are satisfied by any of the following values:

$$x:y:z::\sin^2\tfrac{1}{2}(\beta-\gamma):\sin^2\tfrac{1}{2}(\gamma-a):\sin^2\tfrac{1}{2}(a-\beta)$$
$$::\sin^2\tfrac{1}{2}(\beta-\gamma):\cos^2\tfrac{1}{2}(\gamma-a):\cos^2\tfrac{1}{2}(a-\beta)$$
$$::\cos^2\tfrac{1}{2}(\beta-\gamma):\sin^2\tfrac{1}{2}(\gamma-a):\cos^2\tfrac{1}{2}(a-\beta)$$
$$::\cos^2\tfrac{1}{2}(\beta-\gamma):\cos^2\tfrac{1}{2}(\gamma-a):\sin^2\tfrac{1}{2}(a-\beta).$$

47. If θ_1, θ_2, θ_3, θ_4 are distinct values of θ which satisfy the equation

$$a\cos 2\theta+b\sin 2\theta+c\cos\theta+d\sin\theta+e=0,$$

shew that

$$\frac{a}{\cos s}=\frac{b}{\sin s}=\frac{-c}{\Sigma\cos(s-\theta)}=\frac{-d}{\Sigma\sin(s-\theta)}=\frac{e}{\Sigma\cos\frac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)},$$

where

$$2s=\theta_1+\theta_2+\theta_3+\theta_4.$$

48. Prove that

$$(-1)^{\frac{1}{2}n}\tan^n\theta=1-n\sec\theta\cos\theta+\frac{n(n-1)}{2!}\sec^2\theta\cos 2\theta-\dots\ (n \text{ even}),$$

$$(-1)^{\frac{1}{2}(n-1)}\tan^n\theta=n\sec\theta\sin\theta-\frac{n(n-1)}{2!}\sec^2\theta\sin 2\theta+\dots\ (n \text{ odd}).$$

49. If $\sin^{-1}x=a_1x+a_3x^3+\dots$, shew that the sum of the series

$$a_3x^3+a_9x^9+a_{15}x^{15}+\dots \text{ is } \tfrac{1}{3}\{\cos^{-1}(\sqrt{1+x^2+x^4}-x^2)+\sin^{-1}x\}.$$

50. If α, β, γ... are the n roots of the equation $x^n+p_1x^{n-1}+\dots+p_n=0$, prove that

$$\tan^{-1}\frac{\alpha\sin\theta}{\alpha\cos\theta-x}+\tan^{-1}\frac{\beta\sin\theta}{\beta\cos\theta-x}+\dots$$
$$=\tan^{-1}\frac{p_1\sin\theta\,.\,x^{n-1}+p_2\sin 2\theta\,.\,x^{n-2}+\dots+p_n\sin n\theta}{x^n+p_1\cos\theta\,.\,x^{n-1}+p_2\cos 2\theta\,.\,x^{n-2}+\dots+p_n\cos n\theta}.$$

51. If $(1-c)\tan\theta=(1+c)\tan\phi$, then each of the series

$$c\sin 2\theta-\tfrac{1}{2}c^2\sin 4\theta+\tfrac{1}{3}c^3\sin 6\theta-\dots,$$
$$c\sin 2\phi+\tfrac{1}{2}c^2\sin 4\phi+\tfrac{1}{3}c^3\sin 6\phi+\dots$$

is equal to $\theta-\phi$, where θ and ϕ vanish together, and $c<1$.

52. Prove that

$$\cos\tfrac{1}{3}\pi+\tfrac{1}{2}\cos\tfrac{2}{3}\pi+\tfrac{1}{3}\cos\tfrac{3}{3}\pi+\dots\ \textit{ad inf.}=0.$$

53. Shew that the series

$$\cos x+\frac{1.}{2.3}\cos 3x+\frac{1.3}{2.4.5}\cos 5x+\frac{1.3.5}{2.4.6.7}\cos 7x+\dots$$

assumes the following values,

(1) $\sin^{-1}(\cos\frac{1}{2}x-\sin\frac{1}{2}x)$, when $\pi>x>0$,

(2) $-\sin^{-1}(\cos\frac{1}{2}x+\sin\frac{1}{2}x)$, when $2\pi>x>\pi$.

54. If $c=\cos^2\theta-\frac{1}{3}\cos^3\theta\cos 3\theta+\frac{1}{5}\cos^5\theta\cos 5\theta-\dots,$

shew that $\tan 2c=2\cot^2\theta.$

55. Shew that

$$e^{a\cos\beta}\sin(a\sin\beta)+e^{a\cos 2\beta}\sin(a\sin 2\beta)+\dots e^{a\cos(n-1)\beta}\sin\{a\sin(n-1)\beta\}=0,$$

if $\beta=2\pi/n$.

56. Prove that

$$\sin\theta\,.\,\sin\theta-\tfrac{1}{2}\sin 2\theta\sin^2\theta+\tfrac{1}{3}\sin 3\theta\sin^3\theta-\dots,=\cot^{-1}(1+\cot\theta+\cot^2\theta).$$

57. Prove that

$$\log(\operatorname{cosec} x) = 2\left(\cos^2 x - \tfrac{1}{2}\sin^2 2x + \tfrac{1}{3}\cos^2 3x - \tfrac{1}{4}\sin^2 4x + \ldots\ldots\right).$$

58. Prove that

$$\cos^{-1}(1-x) = \sqrt{2x}\left\{1 + \frac{1}{3}\cdot\frac{1}{2}\left(\frac{x}{2}\right) + \ldots + \frac{1}{2n+1}\cdot\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\left(\frac{x}{2}\right)^n + \ldots\right\}.$$

59. Shew that the sum of the series

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots\ldots \text{ is } \frac{\cos\frac{1}{4}\theta}{\sqrt{2\cos\frac{1}{2}\theta}},$$

where θ lies between $\pm\pi$.

Sum to infinity the series in Examples 60—71.

60. $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots\ldots$

61. $1 - \dfrac{\cos 2\theta}{2!} + \dfrac{\cos 4\theta}{4!} - \ldots\ldots$

62. $\cos\theta + \dfrac{\operatorname{cosec}\theta}{1!}\cos 2\theta + \dfrac{\operatorname{cosec}^2\theta}{2!}\cos 3\theta + \ldots\ldots$

63. $\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \dfrac{1}{2!}\cos 3\theta\cos 4\theta + \dfrac{1}{3!}\cos 4\theta\cos 5\theta + \ldots\ldots$

64. $\sin\theta - \dfrac{1}{3!}\sin 3\theta + \dfrac{1}{5!}\sin 5\theta - \ldots\ldots$

65. $\dfrac{\cos\theta}{1.2.3} + \dfrac{\cos 2\theta}{2.3.4} + \dfrac{\cos 3\theta}{3.4.5} + \ldots\ldots$

66. $\cos a + \dfrac{\cos(a+2\beta)}{3!} + \dfrac{\cos(a+4\beta)}{5!} + \dfrac{\cos(a+6\beta)}{7!} + \ldots\ldots$

67. $\cos\theta\cos\phi - \frac{1}{2}\cos 2\theta\cos 2\phi + \frac{1}{3}\cos 3\theta\cos 3\phi - \ldots\ldots$

68. $\tan a\sin 2x + \dfrac{\tan^2 a\sin 3x}{2!} + \dfrac{\tan^3 a\sin 4x}{3!} + \ldots\ldots$

69. $1 + e^{\cos\theta}\cos(\sin\theta) + \dfrac{e^{2\cos\theta}}{2!}\cos(2\sin\theta) + \dfrac{e^{3\cos\theta}}{3!}\cos(3\sin\theta) + \ldots\ldots$

70. $\sin\theta\,.\,\sin\theta - \frac{1}{2}\sin^2\theta\,.\,\sin 2\theta + \frac{1}{3}\sin^3\theta\,.\,\sin 3\theta - \ldots\ldots$

71. $m\sin^2 a - \frac{1}{2}m^2\sin^2 2a + \frac{1}{3}m^3\sin^2 3a - \ldots\ldots$, where $m < 1$.

CHAPTER XVI.

THE HYPERBOLIC FUNCTIONS.

258. THE hyperbolic cosine, sine, tangent, &c., have already been defined in Chap. XV, by means of the equations

$$\cosh u = \tfrac{1}{2}(e^u + e^{-u}), \quad \sinh u = \tfrac{1}{2}(e^u - e^{-u}), \quad \tanh u = \sinh u/\cosh u,$$

$$\coth u = 1/\tanh u, \quad \operatorname{sech} u = 1/\cosh u, \quad \operatorname{cosech} u = 1/\sinh u,$$

where the exponentials e^u, e^{-u} have their principal values. The hyperbolic functions are expressed in terms of circular functions of iu, by the equations

$$\cosh u = \cos iu, \quad \sinh u = -i \sin iu, \quad \tanh u = -i \tan iu,$$

$$\coth u = i \cot iu, \quad \operatorname{sech} u = \sec iu, \quad \operatorname{cosech} u = i \operatorname{cosec} iu.$$

Relations between the hyperbolic functions.

259. We have, at once from the definitions, the following relations between the hyperbolic functions

$$\cosh^2 u - \sinh^2 u = 1 \quad\text{..........}(1),$$

$$\operatorname{sech}^2 u + \tanh^2 u = 1 \quad\text{..........}(2),$$

$$\coth^2 u - \operatorname{cosech}^2 u = 1 \quad\text{..........}(3).$$

These correspond to the relations

$$\cos^2\theta + \sin^2\theta = 1, \quad \sec^2\theta - \tan^2\theta = 1, \quad \operatorname{cosec}^2\theta - \cot^2\theta = 1,$$

between the circular functions, and are at once deduced from them by putting $\theta = iu$. By means of the relations (1), (2), (3), combined with the definitions, any one hyperbolic function can be expressed in terms of any other one. The results are given in the following table.

	$\sinh u = x$	$\cosh u = x$	$\tanh u = x$	$\coth u = x$	$\operatorname{sech} u = x$	$\operatorname{cosech} u = x$
$\sinh u =$	x	$\sqrt{x^2-1}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{\sqrt{x^2-1}}$	$\frac{\sqrt{1-x^2}}{x}$	$\frac{1}{x}$
$\cosh u =$	$\sqrt{1+x^2}$	x	$\frac{1}{\sqrt{1-x^2}}$	$\frac{x}{\sqrt{x^2-1}}$	$\frac{1}{x}$	$\frac{\sqrt{1+x^2}}{x}$
$\tanh u =$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{\sqrt{x^2-1}}{x}$	x	$\frac{1}{x}$	$\sqrt{1-x^2}$	$\frac{1}{\sqrt{1+x^2}}$
$\coth u =$	$\frac{\sqrt{x^2+1}}{x}$	$\frac{x}{\sqrt{x^2-1}}$	$\frac{1}{x}$	x	$\frac{1}{\sqrt{1-x^2}}$	$\sqrt{1+x^2}$
$\operatorname{sech} u =$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{1}{x}$	$\sqrt{1-x^2}$	$\frac{\sqrt{x^2-1}}{x}$	x	$\frac{x}{\sqrt{1+x^2}}$
$\operatorname{cosech} u =$	$\frac{1}{x}$	$\frac{1}{\sqrt{x^2-1}}$	$\frac{\sqrt{1-x^2}}{x}$	$\sqrt{x^2-1}$	$\frac{x}{\sqrt{1-x^2}}$	x

The addition formulae.

260. We have

$$\cosh(u \pm v) = \cos i(u \pm v) = \cos iu \cos iv \mp \sin iu \sin iv;$$

hence
$$\cosh(u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v \text{.........(4).}$$

Similarly we have

$$\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v \text{.........(5).}$$

These are the addition formulae for the hyperbolic cosine and sine; they may, of course, be verified by substituting the exponential values of the functions. From (4) and (5) we deduce

$$\tanh(u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v} \text{...............(6),}$$

$$\coth(u \pm v) = \frac{\coth u \coth v \pm 1}{\coth v \pm \coth u} \text{...............(7).}$$

261. Since

$$\sinh(u+v) + \sinh(u-v) = 2\sinh u \cosh v,$$
$$\sinh(u+v) - \sinh(u-v) = 2\cosh u \sinh v,$$
$$\cosh(u+v) + \cosh(u-v) = 2\cosh u \cosh v,$$
$$\cosh(u+v) - \cosh(u-v) = 2\sinh u \sinh v,$$

we have, by changing u, v into $\frac{1}{2}(u+v)$, $\frac{1}{2}(u-v)$ respectively,

$$\left.\begin{aligned} \sinh u + \sinh v &= 2\sinh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v)\\ \sinh u - \sinh v &= 2\cosh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v)\\ \cosh u + \cosh v &= 2\cosh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v)\\ \cosh u - \cosh v &= 2\sinh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v) \end{aligned}\right\}\ldots\ldots\ldots(8),$$

which are the formulae for the addition or subtraction of two hyperbolic sines or cosines.

Formulae for multiples and submultiples.

262. From the formulae (4), (5), (6), and (8), the relations between the hyperbolic functions of multiples or submultiples may be deduced, as in the case of the analogous formulae for circular functions. We find

$$\sinh 2u = 2\sinh u\cosh u,$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u = 2\cosh^2 u - 1 = 1 + 2\sinh^2 u,$$

$$\tanh 2u = \frac{2\tanh u}{1+\tanh^2 u},$$

$$\sinh 3u = 3\sinh u + 4\sinh^3 u, \quad \cosh 3u = 4\cosh^3 u - 3\cosh u,$$

$$\tanh 3u = \frac{3\tanh u + \tanh^3 u}{1 + 3\tanh^2 u},$$

$$\cosh\tfrac{1}{2}u = \sqrt{\frac{1+\cosh u}{2}}, \quad \sinh\tfrac{1}{2}u = \sqrt{\frac{\cosh u - 1}{2}},$$

$$\tanh\tfrac{1}{2}u = \sqrt{\frac{\cosh u - 1}{\cosh u + 1}} = \frac{\sinh u}{1+\cosh u}.$$

Series for hyperbolic functions.

263. We have

$$e^u = \cosh u + \sinh u, \quad e^{-u} = \cosh u - \sinh u;$$

thus the series for $\cosh u$, $\sinh u$, in powers of u, are

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \ldots,$$

$$\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \ldots.$$

As in Art. 233, we see that $\cosh u = 1 + R$, $\sinh u = u + S$, where $|R| < \frac{1}{2}|u|^2 e^{|u|}$, $|S| < \frac{1}{6}|u|^3 e^{|u|}$.

Also the principal value of $(\cosh u \pm \sinh u)^m$ is always

$$\cosh mu \pm \sinh mu,$$

whatever m may be; this corresponds to De Moivre's theorem for circular functions. We may express the theorem thus

$$\cosh mu = \tfrac{1}{2}\{(\cosh u + \sinh u)^m + (\cosh u - \sinh u)^m\},$$
$$\sinh mu = \tfrac{1}{2}\{(\cosh u + \sinh u)^m - (\cosh u - \sinh u)^m\}.$$

264. We obtain from the last expressions, by expansion,

$$\sinh mu = m \cosh^{m-1} u \sinh u + \frac{m(m-1)(m-2)}{3!}\cosh^{m-3} u \sinh^3 u + \ldots$$

$$\cosh mu = \cosh^m u + \frac{m(m-1)}{2!}\cosh^{m-2} u \sinh^2 u$$
$$+ \frac{m(m-1)(m-2)(m-3)}{4!}\cosh^{m-4} u \sinh^4 u + \ldots.$$

As in the case of circular functions, we can deduce from these series the expansions of $\sinh mu$, $\cosh mu$ in powers of $\sinh u$; it is however unnecessary to repeat the work of collecting the various coefficients, as we may obtain the result at once by substituting iu for θ in the formula of Art. 214, Chapter XVI. We thus obtain

$$\sinh mu = m \sinh u + \frac{m(m^2-1^2)}{3!}\sinh^3 u$$
$$+ \frac{m(m^2-1^2)(m^2-3^2)}{5!}\sinh^5 u + \ldots,$$

$$\cosh mu = 1 + \frac{m^2}{2!}\sinh^2 u + \frac{m^2(m^2-2^2)}{4!}\sinh^4 u + \ldots,$$

which series hold for all values of m, provided they are convergent, which is the case if $\sinh u \leqq 1$. If we put $\sinh u = 1$, we find

$$u = \log(1+\sqrt{2}).$$

265. From the series for $\sinh mu$ we deduce, as in the case of the circular functions, a series for u in powers of $\sinh u$. Equating the first powers of m, we obtain

$$u = \sinh u - \frac{1}{2}\cdot\frac{1}{3}\sinh^3 u + \frac{1.3}{2.4}\cdot\frac{1}{5}\sinh^5 u - \frac{1.3.5}{2.4.6}\cdot\frac{1}{7}\sinh^7 u + \ldots.$$

This series is convergent if $\sinh u \leqq 1$, or if $u \leqq \log(1+\sqrt{2})$. In particular, we have

$$\log(1+\sqrt{2}) = 1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + \ldots.$$

Periodicity of the hyperbolic functions.

266. The functions $\cosh u$, $\sinh u$ have an imaginary period $2\pi i$, since $e^u = e^{u+2\pi i}$. We have therefore

$$\cosh u = \cosh(u + 2i\pi k), \quad \sinh u = \sinh(u + 2i\pi k),$$

where k is any integer. Since $e^{u+\pi i} = -e^u$, $e^{-(u+\pi i)} = -e^{-u}$, we have $\cosh(u + i\pi) = -\cosh u$, $\sinh(u + i\pi) = -\sinh u$; therefore $\tanh(u + i\pi) = \tanh u$, or the period of $\tanh u$ is $i\pi$, only half that of $\cosh u$, $\sinh u$. We find the following values of $\sinh u$, $\cosh u$, $\tanh u$ corresponding to the arguments 0, $\frac{1}{2}\pi i$, πi, $\frac{3}{2}\pi i$.

	0	$\frac{1}{2}\pi i$	πi	$\frac{3}{2}\pi i$
sinh	0	i	0	$-i$
cosh	1	0	-1	0
tanh	0	$\infty \times i$	0	$\infty \times i$
coth	∞	0	∞	0
sech	1	∞	-1	∞
cosech	∞	$-i$	∞	i

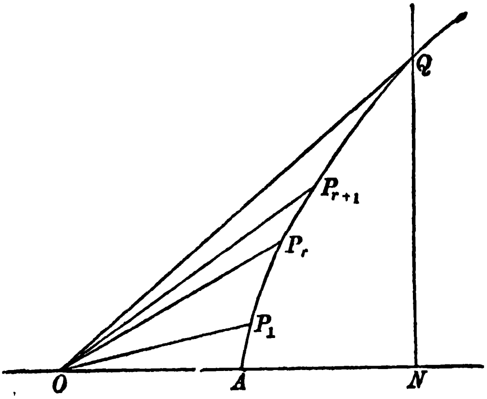
Just as the circular functions are the simplest single periodic functions with a real period, so the hyperbolic functions are the simplest singly periodic functions with an imaginary period.

Area of a sector of a rectangular hyperbola.

267. Let Q be a point on a rectangular hyperbola of semi-transverse-axis a and centre O, and let QN be the ordinate of Q. We have then, from the property of the rectangular hyperbola, $ON^2 - QN^2 = a^2$; if then we let $ON = a\cosh u$, we shall have $NQ = a\sinh u$, where we assume u to be positive or negative according as the ordinate NQ is measured positively or negatively. We proceed to consider the area OAQ bounded by OA, OQ and the arc AQ of the curve. As in the case of a circular sector, considered in Art. 12, we inscribe in the arc AQ an unclosed rectilinear polygon $AP_1P_2P_3\ldots P_r\ldots P_{n-1}Q$, and we define the measure of the area OAQ bounded by OA, OQ, and the arc AQ, as the limit of the measure of the area of the closed polygon

$$OAP_1\ldots P_{n-1}QO,$$

provided this limit exists, when the number of sides of the inscribed polygon is increased indefinitely in such a manner that the limit of the greatest side converges to zero, provided also this limit has a unique value for all sequences of polygons subject to the prescribed condition. Let u_r be the value of u corresponding to the point P_r, and let θ_r denote the circular measure of the angle P_rOA; let u and θ correspond to the point Q.



We have $\tan\theta_r = \tanh u_r$; hence we find

$$\sin\theta_r = \frac{\sinh u_r}{(\cosh 2u_r)^{\frac{1}{2}}}, \text{ and } \cos\theta_r = \frac{\cosh u_r}{(\cosh 2u_r)^{\frac{1}{2}}}.$$

From these values, and the corresponding expressions for $\sin\theta_{r+1}$, $\cos\theta_{r+1}$, we find that

$$\sin(\theta_{r+1} - \theta_r) = \frac{\sinh(u_{r+1} - u_r)}{(\cosh 2u_r \cosh 2u_{r+1})^{\frac{1}{2}}}.$$

Now $$OP_r = a(\cosh^2 u_r + \sinh^2 u_r)^{\frac{1}{2}} = a\cosh^{\frac{1}{2}} 2u_r,$$

and $$OP_{r+1} = a\cosh^{\frac{1}{2}} 2u_{r+1};$$

hence

$$\Delta OP_rP_{r+1} = \tfrac{1}{2} OP_r . OP_{r+1} \sin(\theta_{r+1} - \theta_r) = \tfrac{1}{2} a^2 \sinh(u_{r+1} - u_r).$$

The measure of the area of the rectilineal polygon bounded by OA, OQ and the sides of $AP_1P_2 \ldots P_{n-1}Q$ is therefore

$$\tfrac{1}{2} a^2 \sum_{r=0}^{r=n-1} \sinh(u_{r+1} - u_r),$$

where $u_0 = 0$, $u_n = u$.

This measure is equal to

$$\tfrac{1}{2}a^2 \sum_{r=0}^{r=n-1} \{(u_{r+1} - u_r) + \sigma_r (u_{r+1} - u_r)^3 e^{u_{r+1} - u_r}\},$$

in virtue of a theorem proved in Art. 263, where all the numbers a_r are less than 1/6.

The length of the side $P_r P_{r+1}$ is

$$a\{(\cosh u_{r+1} - \cosh u_r)^2 + (\sinh u_{r+1} - \sinh u_r)^2\}^{\frac{1}{2}},$$

which reduces to

$$2a \cosh^{\frac{1}{2}} (u_r + u_{r+1}) \sinh \tfrac{1}{2} (u_{r+1} - u_r).$$

Also $u_{r+1} - u_r < \sinh (u_{r+1} - u_r)$; therefore the ratio

$$(u_{r+1} - u_r)/P_r P_{r+1}$$

is $\quad < a^{-1} \cosh \tfrac{1}{2}(u_{r+1} - u_r)/\cosh^{\frac{1}{2}} (u_r + u_{r+1}) < a^{-1} \cosh \tfrac{1}{2} u.$

Since now $(u_{r+1} - u_r)/P_r P_{r+1}$ is less than a fixed number independent of r and of the particular polygon, we see that in any sequence of polygons the greatest of the numbers $u_{r+1} - u_r$ in one of the polygons converges to zero as the greatest of the sides $P_r P_{r+1}$ does so. In the polygon we may therefore suppose

$$u_{r+1} - u_r < \eta_n,$$

for all values of r, where η_n converges to zero as the number of sides is indefinitely increased.

We now see that the measure of the area of the rectilineal polygon differs from $\frac{1}{2}a^2 \sum\limits_{r=0}^{n-1} (u_{r+1} - u_r)$, or $\frac{1}{2}a^2 u$, by a number less than

$$\tfrac{1}{12} a^2 \eta_n^2 e^{\eta_n} \sum_{r=0}^{n-1} (u_{r+1} - u_r) \quad \text{or} \quad \tfrac{1}{12} a^2 \eta_n^2 e^{\eta_n} u;$$

and this converges to zero when η_n does so. It has now been proved that $\frac{1}{2}a^2 u$ is the unique limit of the measure of the areas of the rectilineal polygons in any sequence subject to the prescribed condition. Therefore the area of the sector OAQ bounded by OA, OQ and the arc AQ of the rectangular hyperbola is $\frac{1}{2}a^2 u$.

The area of any sector of which the extremities are represented by u, u' is clearly measured by $\frac{1}{2}a^2 (u \sim u')$.

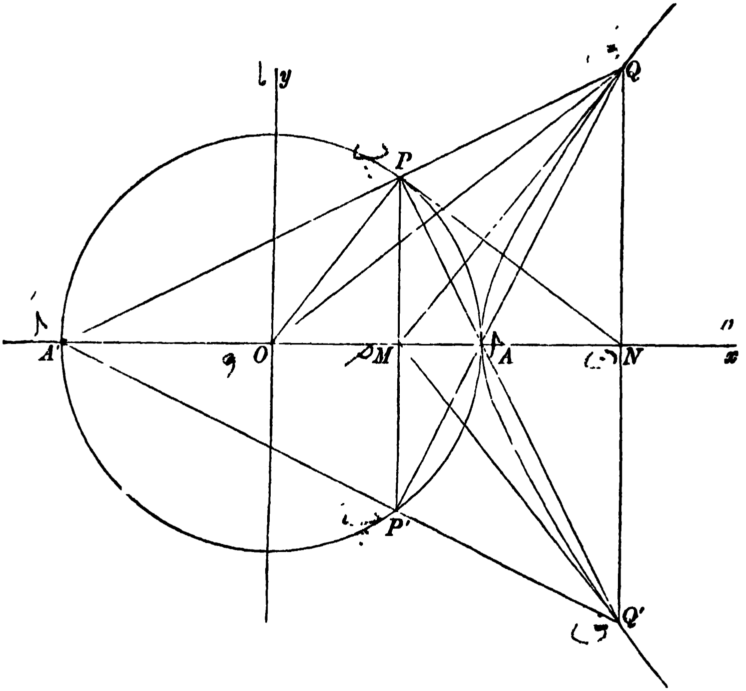
It should be observed that, to represent points on the other branch of the rectangular hyperbola, u must be changed into $i\pi - u$, since

$$\cosh (i\pi - u) = -\cosh u,$$

and $\qquad \sinh (i\pi - u) = \sinh u.$

268. If we describe a circle[1] of radius $OA = a$, and let P be any point on the circle, MP its ordinate, then denoting the angle POA by θ, we have area $OAP = \frac{1}{2}a^2\theta$. Let PN be the tangent at P, we have then

$$OM = a\cos\theta,\quad MP = a\sin\theta,\quad NP = a\tan\theta,\quad MA = a\operatorname{vers}\theta.$$



From N draw NQ perpendicular to OA and equal to NP, then $ON^2 - NQ^2 = a^2$; therefore the locus of Q is a rectangular hyperbola of semi-axis a. Now denote the area of the sector OAQ by $\frac{1}{2}a^2u$, then as we have proved in the last Article, we have $ON = a\cosh u$, $NQ = a\sinh u$. Thus we see that, just as the ordinate and abscissa of a point P on the circle are denoted by $a\sin\theta$, $a\cos\theta$, respectively, where $\frac{1}{2}a^2\theta$ is the area of the circular sector OAP, so the ordinate and abscissa of the point Q on the rectangular hyperbola are denoted by $a\sinh u$, $a\cosh u$, respectively, where $\frac{1}{2}a^2u$ is the area of the sector OAQ Thus the hyperbolic sine and cosine have a property in reference to the rectangular hyperbola, exactly analogous to that of the sine and cosine with reference to the circle. For this reason the former functions are called hyperbolic functions, just as the latter are called circular functions.

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[1] The figure in this Article is taken from a tract by Greenhill entitled "A Chapter on the Integral Calculus."

269. We have, from the figure of the last Article, when we consider the point Q on the rectangular hyperbola, corresponding to the point P on the circle,

$$a \tan\theta = NQ = a \sinh u, \quad \text{and} \quad a \sec\theta = ON = a\cosh u;$$

therefore the arguments θ, u, for corresponding points, satisfy the relations $\tan\theta = \sinh u$, $\sec\theta = \cosh u$. Since

$$\tanh \tfrac{1}{2}u = \frac{\sinh u}{1+\cosh u},$$

we have $$\tanh\tfrac{1}{2}u = \frac{\tan\theta}{1+\sec\theta} = \frac{\sin\theta}{1+\cos\theta} = \tan\tfrac{1}{2}\theta,$$

or θ and u satisfy the relation $\tanh\frac{1}{2}u = \tan\frac{1}{2}\theta$.

Since $\Delta OQM <$ sector $OAQ < \Delta OAQ$, we have

$$\tanh u < u < \sinh u.$$

It follows that the limits of $\frac{\tanh u}{u}$, $\frac{\sinh u}{u}$, when u is indefinitely diminished, are each unity, since $\cosh 0 = 1$.

270. We have

$$e^u = \cosh u + \sinh u = \sec\theta + \tan\theta;$$

therefore $$u = \log_e(\sec\theta + \tan\theta) = \log_e \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta).$$

Various names have been given to the argument θ; it is called by Cayley the *Gudermannian* function of u, and denoted by $gd\,u$, so that $\theta = gd\,u$, $u = gd^{-1}\theta = \log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$; this name was given in honour of Gudermann, who however called the function[1] the *longitude* of u. By Lambert, θ was called the *transcendent angle*, and by Hoüel[2] the *hyperbolic amplitude* of u (written amh u). A table of the values of $\log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$ for values of θ from 0° to 90° at intervals of 30′, and to 12 places of decimals, is to be found in Legendre's *Théorie des Fonctions Elliptiques*, Vol. II. Table IV. The table which we give at the end of the Chapter, for intervals of one degree, was extracted[3] from Legendre's table by Prof. Cayley.

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[1] See *Crelle's Journal* for 1833.

[2] See "Théorie des Fonctions complexes."

[3] See the *Quarterly Journal*, Vol. XX. p. 220.

The table enables us to find the numerical values of the hyperbolic functions of u, by means of the relations

$$\sinh u = \tan\theta, \quad \cosh u = \sec\theta,$$

using a table of natural tangents or secants of angles.

Those who desire further information on the subject of the hyperbolic functions and their applications, may refer to Laisant's "Essai sur les Fonctions Hyperboliques" in the *Mémoires de la Société des Sciences de Bordeaux*, Vol. x., also the treatises "Die hyperbolischen Functionen" by E. Heis, and "Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbol-funktionen" by Günther.

Expressions for the circular functions of complex arguments.

271. The circular functions with a complex argument may, by the use of the notation of the hyperbolic functions, be conveniently expressed in the form $\alpha + i\beta$, where α and β are real quantities.

Thus $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$;

hence $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y \quad \text{.........(9)}.$

Similarly we find

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y \quad \text{......(10)}.$$

Also
$$\tan(x+iy) = \frac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$
$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$$

hence

$$\tan(x+iy) = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} \quad \text{..............(11)}.$$

The inverse circular functions of complex arguments.

272. We shall first consider the function $\sin^{-1}(x+iy)$. Let $\sin^{-1}(x+iy) = \alpha + i\beta$, then

$$x + iy = \sin(\alpha + i\beta) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta,$$

or $x = \sin\alpha\cosh\beta$, $y = \cos\alpha\sinh\beta$; we have therefore, for the determination of β, the equation $x^2/\cosh^2\beta + y^2/\sinh^2\beta = 1$, or $x^2(\cosh^2\beta - 1) + y^2\cosh^2\beta = \cosh^2\beta(\cosh^2\beta - 1)$.

If we solve this quadratic for $\cosh^2\beta$, we find

$$\cosh^2\beta = \tfrac{1}{2}(x^2+y^2+1) \pm \tfrac{1}{2}\sqrt{(x^2+y^2+1)^2-4x^2};$$

therefore $\cosh\beta = \pm\tfrac{1}{2}\sqrt{x^2+y^2+2x+1} \pm \tfrac{1}{2}\sqrt{x^2+y^2-2x+1}$,

and since $\cosh\beta$ is positive, we must have, if x is positive,

$$\cosh\beta = \tfrac{1}{2}\sqrt{(x+1)^2+y^2} \pm \tfrac{1}{2}\sqrt{(x-1)^2+y^2}.$$

The corresponding value of $\sin\alpha$ is

$$x/\cosh\beta \text{ or } \tfrac{1}{2}\sqrt{(x+1)^2+y^2} \mp \tfrac{1}{2}\sqrt{(x-1)^2+y^2};$$

now $\cosh\beta > 1 > \sin\alpha$, hence we have

$$\cosh\beta = \tfrac{1}{2}\sqrt{(x+1)^2+y^2} + \tfrac{1}{2}\sqrt{(x-1)^2+y^2} = u,$$
$$\sin\alpha = \tfrac{1}{2}\sqrt{(x+1)^2+y^2} - \tfrac{1}{2}\sqrt{(x-1)^2+y^2} = v.$$

These are the values of $\cosh\beta$, $\sin\alpha$, whether x is positive or negative.

The quadratic $\cosh\beta = u$ gives $\beta = \pm\log\{u+\sqrt{u^2-1}\}$; we have therefore

$$\sin^{-1}(x+iy) = k\pi + (-1)^k\sin^{-1}v \pm i\log\{u+\sqrt{u^2-1}\},$$

where k is an integer, and $\sin^{-1}v$ is the principal value of α, which satisfies the condition $\sin\alpha = v$. To determine the ambiguous sign, put $x=0$, then $\sin^{-1}iy = k\pi \pm i\log(\sqrt{1+y^2}+y)$; hence

$$iy = \pm\cos k\pi\sin[i\log(\sqrt{1+y^2}+y)]$$
$$= \pm(-1)^k\frac{1}{2i}\left\{\frac{1}{y+\sqrt{y^2+1}} - y - \sqrt{y^2+1}\right\} = \pm(-1)^k iy,$$

hence the ambiguous sign must be that of $(-1)^k$, or

$$\sin^{-1}(x+iy) = k\pi + (-1)^k\sin^{-1}v + (-1)^k i\log\{u+\sqrt{u^2-1}\} \ldots (12),$$

where $u = \tfrac{1}{2}\sqrt{(x+1)^2+y^2} + \tfrac{1}{2}\sqrt{(x-1)^2+y^2}$,

and $v = \tfrac{1}{2}\sqrt{(x+1)^2+y^2} - \tfrac{1}{2}\sqrt{(x-1)^2+y^2}$.

If we consider $\sin^{-1}v + i\log\{u+\sqrt{u^2-1}\}$ as the principal value of $\sin^{-1}(x+iy)$, and denote it by $\sin^{-1}(x+iy)$, the general value is $k\pi+(-1)^k\sin^{-1}(x+iy)$, which is the same expression as for real arguments.

A special case is that of $x>1$, $y=0$; in this case $u=x$, $v=1$, and the principal value of $\sin^{-1}x$ is $\tfrac{1}{2}\pi + i\log\{x+\sqrt{x^2-1}\}$. We know *a priori* that $\sin^{-1}x$ can have no real value when $x>1$.

273. Next let $\cos^{-1}(x+iy)=\alpha+i\beta$, we have then, as in the last case, $x=\cos\alpha\cosh\beta$, $y=-\sin\alpha\sinh\beta$, and we find, as before,

$$\cosh\beta=\tfrac{1}{2}\sqrt{(x+1)^2+y^2}+\tfrac{1}{2}\sqrt{(x-1)^2+y^2}=u,$$

$$\cos\alpha=\tfrac{1}{2}\sqrt{(x+1)^2+y^2}-\tfrac{1}{2}\sqrt{(x-1)^2+y^2}=v\,;$$

hence $\cos^{-1}(x+iy)=2k\pi\pm\cos^{-1}v\pm i\log\{u+\sqrt{u^2-1}\}$.

To determine the sign of the last term, we put $x=0$, then

$$iy=\cos\left[\pm\tfrac{1}{2}\pi\pm i\log\left(y+\sqrt{y^2+1}\right)\right]=\mp\sin\left\{\pm i\log\left(y+\sqrt{y^2+1}\right)\right\}$$
$$=(\mp)(\pm iy)\,;$$

hence we see that the second ambiguous sign must be the opposite of the first, or

$$\cos^{-1}(x+iy)=2k\pi\pm\left\{\cos^{-1}v-i\log\left(u+\sqrt{u^2-1}\right)\right\}\ldots(13).$$

If $\cos^{-1}v-i\log\left(u+\sqrt{u^2-1}\right)$ denotes the principal value of $\cos^{-1}(x+iy)$, then the general value is $2k\pi\pm\cos^{-1}(x+iy)$.

274. Let $\tan^{-1}(x+iy)=\alpha+i\beta$, then

$$x+iy=\frac{\sin 2\alpha+i\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta},$$

hence $$x=\frac{\sin 2\alpha}{\cos 2\alpha+\cosh 2\beta},\quad y=\frac{\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}\,;$$

we have

$$x^2+y^2=\frac{\sin^2 2\alpha+\sinh^2 2\beta}{(\cos 2\alpha+\cosh 2\beta)^2}=\frac{\cosh^2 2\beta-\cos^2 2\alpha}{(\cos 2\alpha+\cosh 2\beta)^2}=\frac{\cosh 2\beta-\cos 2\alpha}{\cosh 2\beta+\cos 2\alpha},$$

or $$1-x^2-y^2=\frac{2\cos 2\alpha}{\cosh 2\beta+\cos 2\alpha},\text{ and }1+x^2+y^2=\frac{2\cosh 2\beta}{\cosh 2\beta+\cos 2\alpha}\,;$$

therefore $$\tan 2\alpha=\frac{2x}{1-x^2-y^2},\text{ and }\tanh 2\beta=\frac{2y}{1+x^2+y^2}.$$

Since $\dfrac{e^{2\beta}-e^{-2\beta}}{e^{2\beta}+e^{-2\beta}}=\dfrac{2y}{1+x^2+y^2}$, we have $e^{4\beta}=\dfrac{x^2+(y+1)^2}{x^2+(y-1)^2}$,

or $$\beta=\tfrac{1}{4}\log\left\{\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right\},$$

hence the values of $\tan^{-1}(x+iy)$ are given by

$$\tan^{-1}(x+iy)=k\pi+\tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}+\tfrac{1}{4}i\log\left\{\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right\}\ldots(14).$$

The inverse hyperbolic functions.

275. If $\sinh\alpha=z$, then α is called the *inverse hyperbolic sine* of z, and is denoted by $\sinh^{-1}z$. A similar definition applies to $\cosh^{-1}z$ and $\tanh^{-1}z$.

If $z = \sinh \alpha = -i \sin i\alpha$, we have $iz = \sin i\alpha$, or $\alpha = \frac{1}{i} \sin^{-1}(iz)$. Similarly if $z = \cosh \alpha = \cos i\alpha$, we have $\alpha = \frac{1}{i} \cos^{-1} z$; we find also if $z = \tanh \alpha$, $\alpha = \frac{1}{i} \tan^{-1}(iz)$. We have therefore the inverse hyperbolic functions expressed as inverse circular functions by the equations

$$\sinh^{-1} z = -i \sin^{-1}(iz),$$
$$\cosh^{-1} z = -i \cos^{-1}(z),$$
$$\tanh^{-1} z = -i \tan^{-1}(iz).$$

276. By means of the expressions we have found for the inverse circular functions of a complex argument, we may find the values of the inverse hyperbolic functions. We shall however find the expressions for them independently.

(1) If $z = \sinh \alpha$, we have $e^{\alpha} - e^{-\alpha} = 2z$; solving this as a quadratic for e^{α}, we find $e^{\alpha} = z \pm \sqrt{1+z^2}$,
hence $\alpha = 2ik\pi + \log_e(z + \sqrt{1+z^2})$ or $2ik\pi + \log_e(z - \sqrt{1+z^2})$,
both values of α are included in the expression

$$ik\pi + (-1)^k \log(z + \sqrt{1+z^2}).$$

Thus the general value of $\sinh^{-1} y$ is $ik\pi + (-1)^k \log_e(z + \sqrt{1+z^2})$, and its principal value is $\log_e(z + \sqrt{1+z^2})$; this principal value is the one which is usually denoted by $\sinh^{-1} z$.

(2) If $z = \cosh \alpha$, we have $e^{\alpha} + e^{-\alpha} = 2z$; hence we find

$$e^{\alpha} = z \pm \sqrt{z^2 - 1}, \text{ thus } \alpha = 2ik\pi \pm \log_e(z + \sqrt{z^2-1}),$$

hence $2ik\pi \pm \log_e(z + \sqrt{z^2-1})$ is the general value of $\cosh^{-1} z$; the principal value, which is the one generally understood to be denoted by $\cosh^{-1} z$, is $\log_e(z + \sqrt{z^2-1})$.

(3) If $z = \tanh \alpha$, we have $\frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} = z$, or $e^{2\alpha} = \frac{1+z}{1-z}$, hence $\alpha = ik\pi + \frac{1}{2} \log_e\left(\frac{1+z}{1-z}\right)$; this is the general value of $\tanh^{-1} z$, the principal value being $\frac{1}{2} \log_e\left(\frac{1+z}{1-z}\right)$.

(4) We find for the principal values of $\coth^{-1} z$, $\operatorname{sech}^{-1} z$, $\operatorname{cosech}^{-1} z$, the expressions

$$\frac{1}{2} \log_e\left(\frac{z+1}{z-1}\right), \quad \log_e \frac{1 + \sqrt{1-z^2}}{z}, \quad \log_e \frac{1 + \sqrt{1+z^2}}{z}$$

respectively.

The solution of cubic equations.

277. We have shewn, in Art. 117, that when the roots of the cubic $x^3+qx+r=0$ are all real, and q is negative, they are $\sqrt{-\frac{4}{3}q}\sin\theta$, $\sqrt{-\frac{4}{3}q}\sin(\theta+\frac{2}{3}\pi)$, $\sqrt{-\frac{4}{3}q}\sin(\theta+\frac{4}{3}\pi)$, where $\sin 3\theta=\left(-\frac{27r^2}{4q^3}\right)^{\frac{1}{2}}$. We shall now shew how to solve the cubic in the case when two of the roots are imaginary. In this case, the condition $27r^2+4q^3>0$ is satisfied.

(1) Suppose q positive; consider the cubic

$$4\sinh^3 u+3\sinh u=\sinh 3u,$$

let $x=a\sinh u$, then x satisfies the equation

$$x^3+\tfrac{3}{4}a^2 . x-\tfrac{1}{4}a^3\sinh 3u=0;$$

this will coincide with the cubic $x^3+qx+r=0$, if $q=\frac{3}{4}a^2$, $r=-\frac{1}{4}a^3\sinh 3u$, or $\sinh 3u=-4\left(\frac{27}{64}\frac{r^2}{q^3}\right)^{\frac{1}{2}}$.

Now the roots of the cubic $4\sinh^3 u+3\sinh u=\sinh 3u$ are $\sinh u$, $\sinh(u+\frac{2}{3}\pi i)$ and $\sinh(u+\frac{4}{3}\pi i)$, hence the roots of the cubic $x^3+qx+r=0$ are

$$\sqrt{\tfrac{4}{3}q}\sinh u,\ \sqrt{\tfrac{4}{3}q}\sinh(u+\tfrac{2}{3}\pi i),\ \sqrt{\tfrac{4}{3}q}\sinh(u+\tfrac{4}{3}\pi i),$$

or

$$\sqrt{\tfrac{4}{3}q}\sinh u,\ \sqrt{\tfrac{1}{3}q}(-\sinh u\pm i\sqrt{3}\cosh u),$$

where $\sinh 3u=-\frac{1}{2}\left(27\frac{r^2}{q^3}\right)^{\frac{1}{2}}$. We find the number $3u$ from a table of hyperbolic sines, when the numerical values of q and r are given, and then $\sinh u$, $\cosh u$ from the same tables; thus the numerical values of the roots will be found.

(2) When q is negative; consider the equation

$$4\cosh^3 u-3\cosh u=\cosh 3u,$$

we find, as in the last case, that if $q=-\frac{3}{4}a^2$, $r=-\frac{1}{4}a^3\cosh 3u$, the cubic which $a\cosh u$ satisfies is $x^3+qx+r=0$; thus the roots required are

$$\sqrt{-\tfrac{4}{3}q}\cosh u,\ \sqrt{-\tfrac{4}{3}q}\cosh(u+\tfrac{2}{3}\pi i),\ \sqrt{-\tfrac{4}{3}q}\cosh(u+\tfrac{4}{3}\pi i),$$

or

$$\sqrt{-\tfrac{4}{3}q}\cosh u,\ \sqrt{-\tfrac{1}{3}q}(-\cosh u\pm\sqrt{3}\sinh u),$$

where $\cosh 3u=-\frac{1}{2}\left(-27\frac{r^2}{q^3}\right)^{\frac{1}{2}}$. Hence, as in the last case, we can employ tables of hyperbolic functions to find the numerical values of the roots of the cubic, when the values of q and r are given.

278. *Table of values of* u *for given values of θ.*

θ		$u=\log_e \tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$	θ		$u=\log_e \tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$
0°	·0	·0	46°	·8028515	·9062755
1°	·0174533	·0174542	47°	·8203047	·9316316
2°	·0349066	·0349137	48°	·8377580	·9574669
3°	·0523599	·0523838	49°	·8552113	·9838079
4°	·0698132	·0698699	50°	·8726646	1·0106832
5°	·0872665	·0873774	51°	·8901179	1·0381235
6°	·1047198	·1049117	52°	·9075712	1·0661617
7°	·1221730	·1224781	53°	·9250245	1·0948335
8°	·1396263	·1400822	54°	·9424778	1·1241772
9°	·1570796	·1577296	55°	·9509311	1·1542346
10°	·1745329	·1754258	56°	·9773844	1·1850507
11°	·1919862	·1931766	57°	·9948377	1·2166748
12°	·2094395	·2109867	58°	1·0122910	1·2491606
13°	·2268928	·2288650	59°	1·0297443	1·2825668
14°	·2443461	·2468145	60°	1·0471976	1·3169579
15°	·2617994	·2648422	61°	1·0646508	1·3524048
16°	·2792527	·2829545	62°	1·0821041	1·3889860
17°	·2967060	·3011577	63°	1·0995574	1·4267882
18°	·3141593	·3194583	64°	1·1170107	1·4659083
19°	·3316126	·3378629	65°	1·1344640	1·5064542
20°	·3490659	·3563785	66°	1·1519173	1·5485472
21°	·3665191	·3750121	67°	1·1693706	1·5923237
22°	·3839724	·3937710	68°	1·1868239	1·6379387
23°	·4014257	·4126626	69°	1·2042772	1·6855685
24°	·4188790	·4316947	70°	1·2217305	1·7354152
25°	·4363323	·4508753	71°	1·2391838	1·7877120
26°	·4537856	·4702127	72°	1·2566371	1·8427300
27°	·4712389	·4897154	73°	1·2740904	1·9007867
28°	·4886922	·5093923	74°	1·2915436	1·9622572
29°	·5061455	·5292527	75°	1·3089969	2·0275894
30°	·5235988	·5493061	76°	1·3264502	2·0973240
31°	·5410521	·5695627	77°	1·3439035	2·1721218
32°	·5585054	·5900329	78°	1·3613568	2·2528027
33°	·5759587	·6107275	79°	1·3788101	2·3404007
34°	·5934119	·6316581	80°	1·3962634	2·4362460
35°	·6108652	·6528366	81°	1·4137167	2·5420904
36°	·6283185	·6742755	82°	1·4311700	2·6603061
37°	·6457718	·6959880	83°	1·4486233	2·7942190
38°	·6632251	·7179880	84°	1·4660766	2·9487002
39°	·6806784	·7402901	85°	1·4835299	3·1313013
40°	·6981317	·7629095	86°	1·5009832	3·3546735
41°	·7155850	·7858630	87°	1·5184364	3·6425334
42°	·7330383	·8091672	88°	1·5358897	4·0481254
43°	·7504916	·8328406	89°	1·5533430	4·7413488
44°	·7679449	·8569026	90°	1·5707963	∞
45°	·7853982	·8813736			

EXAMPLES ON CHAPTER XVI.

1 Prove that

$$8\sinh nx\sinh^2 x=2\sinh(n+2)x-4\sinh nx+2\sinh(n-2)x.$$

2. If $\cos(\alpha+i\beta)=\cos\phi+i\sin\phi$, shew that $\sin\phi=\pm\sin^2\alpha=\pm\sinh^2\beta$.

3. If $\cos(\theta+i\phi)\cos(\alpha+i\beta)=1$, prove that $\tanh^2\phi\cosh^2\beta=\sin^2\alpha$,
and $$\tanh^2\beta\cosh^2\phi=\sin^2\theta.$$

4. If $$\tan y=\tan\alpha\tanh\beta,\quad \tan z=\cot\alpha\tanh\beta,$$
shew that $$\tan(y+z)=\sinh 2\beta\operatorname{cosec}2\alpha.$$

5. Reduce $e^{\sin(\alpha+i\beta)}$ to the form $A+iB$.

6. If $$\log_e\sin(\theta+i\phi)=\alpha+i\beta,$$
shew that $$2\cos 2\theta=2\cosh 2\phi-4e^{2\alpha},$$
and $$\cos(\theta-\beta)=e^{2\phi}\cos(\theta+\beta).$$

7. If $\tan(x+iy)=\sin(u+iv)$, shew that $\coth v\sinh 2y=\cot u\sin 2x$.

8. Express $\{\cos(\theta+i\phi)+i\sin(\theta-i\phi)\}^{\alpha+i\beta}$ in the form $A+iB$.

9. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta+\tanh 2\phi}{\tan 2\theta-\tanh 2\phi}\right)+\tan^{-1}\left(\frac{\tan\theta-\tanh\phi}{\tan\theta+\tanh\phi}\right)=\tan^{-1}(\cot\theta\coth\phi).$$

10. If $$u=\cos\alpha-\tfrac{1}{3}\cos 3\alpha+\tfrac{1}{5}\cos 5\alpha-\ldots\ldots,$$
$$v=\sin\alpha-\tfrac{1}{3}\sin 3\alpha+\tfrac{1}{5}\sin 5\alpha\quad\ldots\ldots,$$
prove that $u=\frac{1}{4}\pi$, when $0\leqq\alpha<\frac{1}{2}\pi$ and $\cosh 2v=\sec\alpha$.

11. Prove that the sum of the infinite series

$$1+\frac{\cos 4\theta}{4!}+\frac{\cos 8\theta}{8!}+\frac{\cos 12\theta}{12!}+\ldots\ldots$$

is $$\tfrac{1}{2}\{\cos(\cos\theta)\cosh(\sin\theta)+\cos(\sin\theta)\cosh(\cos\theta)\}.$$

12. Prove that

$$\sum_{n=0}^{n=\infty}\frac{(-1)^n}{(2n)!}\frac{\sin(2m+1)n\theta}{\sin n\theta}=2\sum_{p=1}^{p=m}\{\cos(\cos p\theta)\cosh(\sin p\theta)\}+\cos\alpha,$$

where α is the unit of circular measure.

13. From Euler's theorem

$$\frac{\sin x}{x}=\cos\tfrac{1}{2}x\cos\tfrac{1}{4}x\cos\tfrac{1}{8}x\ldots\ldots$$

deduce that

(1) $$\frac{1}{\log_e x}=\frac{1}{x-1}+\frac{1}{2}\frac{1}{1+x^{\frac{1}{2}}}+\frac{1}{4}\frac{1}{1+x^{\frac{1}{4}}}+\frac{1}{8}\frac{1}{1+x^{\frac{1}{8}}}+\ldots\ldots$$

(2) $$\frac{1}{x^2}=\operatorname{cosech}^2 x+\frac{1}{2^2}\operatorname{sech}^2\frac{1}{2}x+\frac{1}{4^2}\operatorname{sech}^2\frac{1}{4}x+\frac{1}{8^2}\operatorname{sech}^2\frac{1}{8}x+\ldots\ldots$$

CHAPTER XVII.

INFINITE PRODUCTS.

The convergence of infinite products.

279. LET $z_1, z_2, \dots z_n, \dots$ be a sequence of real or complex numbers formed according to any prescribed law, and consider the product $P_n \equiv z_1 z_2 \dots z_n$ of the first n of these numbers.

If P_n converges to a definite limit P, different from zero, as n is indefinitely increased, P is said to be the limit, or limiting value, of the infinite product $z_1 z_2 z_3 \dots z_n \dots$, and that infinite product is said to be convergent.

It is convenient to exclude the case of those products for which P_n converges to zero from the class of convergent infinite products.

If $P_n = |P_n|(\cos\theta_n + i\sin\theta_n)$, where $|P_n|$ denotes the modulus of P_n, it is necessary and sufficient for the convergence of the infinite product that both $|P_n|$ and θ_n should converge to definite values as n is indefinitely increased. In case $|P_n|$ increases indefinitely, as n is indefinitely increased, the infinite product is said to be divergent. In other cases in which the product is not convergent it is said to oscillate, but oscillating products are frequently spoken of as divergent.

The necessary and sufficient condition that the infinite product $z_1 z_2 \dots z_n \dots$ should converge to a definite value (other than zero) is that, corresponding to each arbitrarily chosen positive number ϵ, an integer n can be so chosen that $|z_{n+1} z_{n+2} \dots z_{n+r} - 1| < \epsilon$, for all values $1, 2, 3, \dots$ of r. To shew that this condition is necessary, let us assume that P_n converges to P, a number different from zero. All except a finite set of the numbers $|P_1|, |P_2|, \dots |P_n| \dots$ are greater than $|P| - \eta$, where η is an arbitrarily chosen positive number such that $|P| - \eta > 0$; also none of these vanishes, therefore there exists a positive number k which is less than all the numbers $|P_1|, |P_2|, \ldots |P_n| \ldots$. Since P_n converges to a definite limit, n may be so chosen, corresponding to ϵ, that $|P_{n+r} - P_n| < k\epsilon$, for $r = 1, 2, 3, \ldots$.

Hence we have $|z_{n+1}z_{n+2} \ldots z_{n+r} - 1| < k\epsilon / |z_1 z_2 \ldots z_n| < \epsilon$, and therefore the condition stated is necessary.

To shew that the condition is sufficient, let us assume it to hold. For an assigned value of ϵ, n can be so fixed that $z_{n+1}z_{n+2} \ldots z_{n+r} = 1 + \rho_{n,r}$, where $|\rho_{n,r}| < \epsilon$, for $r = 1, 2, 3, \ldots$. We have then $P_{n+r} = P_n(1 + \rho_{n,r})$, and therefore $|P_{n+r}| < |P_n|(1 + \epsilon)$, for all positive integral values of r; it follows that all the numbers $|P_1|, |P_2|, \ldots |P_n| \ldots$ are less than a fixed positive number λ. From $|z_{n+1}z_{n+2} \ldots z_{n+r} - 1| < \epsilon$, we have $|P_{n+r} - P_n| < \lambda\epsilon$, for $r = 1, 2, 3, \ldots$, and since $\lambda\epsilon$ may be chosen as small as we please by choosing ϵ small enough, we see that P_n must converge to a definite limit.

A convenient method of considering the convergence of the infinite product $z_1 z_2 \ldots z_n \ldots$, is to consider the series

$$\log_e z_1 + \log_e z_2 + \ldots + \log_e z_n + \ldots.$$

If this series is convergent the infinite product converges to a value other than zero, and conversely. If the infinite product converges to zero, the series diverges to $-\infty$, and for this reason, as before, we exclude this case.

To prove that the convergence of the infinite series and of the infinite product are equivalent, we observe that the necessary and sufficient condition for the convergence of the series is that n can be so determined, for each ϵ, that $|\log_e(z_{n+1}z_{n+2} \ldots z_{n+r})|$ or $|\log_e(1 + \rho_{n,r})| < \epsilon$, for $r = 1, 2, 3, \ldots$.

If this condition is satisfied, we have, on employing the theorem $|e^z - 1| < |z|(1 + \frac{1}{2}|z|e^{|z|})$ established in Art. 230 (1), $|\rho_{n,r}| < \epsilon(1 + \frac{1}{2}\epsilon e^{\epsilon})$. If now η be an arbitrarily chosen positive number, ϵ can be so chosen that $\epsilon(1 + \frac{1}{2}\epsilon e^{\epsilon}) < \eta$, and thus n can be so chosen that $|\rho_{n,r}|$ or $|z_{n+1}z_{n+2} \ldots z_{n+r} - 1|$ is $< \eta$, for $r = 1, 2, 3, \ldots$; therefore the infinite product is convergent. Conversely let us assume that n can be so chosen that $|\rho_{n,r}| < \epsilon$, for $r = 1, 2, 3, \ldots$. It has been shewn in Art. 249 (1) that if $|z| < 1$,

$$|\log_e(1 + z)| < |z|\left(1 + \frac{1}{2}\frac{|z|}{1 - |z|}\right),$$

therefore $$|\log_e(1 + \rho_{n,r})| < \epsilon\left(1 + \frac{1}{2}\frac{\epsilon}{1 - \epsilon}\right);$$

or $\log_e(z_{n+1}z_{n+2}\dots z_{n+r}) < \eta$, provided $\epsilon\left(1+\frac{1}{2}\frac{\epsilon}{1-\epsilon}\right) < \eta$, and if η is prescribed, ϵ can be so determined as to satisfy this condition. Therefore the condition of convergence of the series is satisfied.

280. Suppose $u_1, u_2, \dots u_n, \dots$ to be a sequence of real positive numbers each of which is less than 1; it will be shewn that the infinite products

$$(1+u_1)(1+u_2)\dots(1+u_n)\dots \text{ or } \prod_1^\infty(1+u)$$

and

$$(1-u_1)(1-u_2)\dots(1-u_n)\dots \text{ or } \prod_1^\infty(1-u)$$

both converge, or not, according as the series $u_1+u_2+\dots+u_n+\dots$ is convergent or divergent.

Since

$$(1+u_1)(1+u_2)\dots(1+u_n) > 1+u_1+u_2+\dots+u_n,$$

it is clear that the product $\Pi(1+u)$ diverges if the series $u_1+u_2+\dots$ does so. Also

$$\frac{1}{(1-u_1)(1-u_2)\dots(1-u_n)} > (1+u_1)(1+u_2)\dots(1+u_n),$$

hence if Σu diverges the product $(1-u_1)(1-u_2)\dots(1-u_n)$ converges to zero, and is therefore considered as non-convergent.

Next, if Σu converges, let ϵ be an arbitrarily chosen positive number less than 1, then n can be so chosen that

$$u_{n+1}+u_{n+2}+\dots+u_{n+r} < \epsilon,$$

for $r=1, 2, 3, \dots$. We have, as in Art. 226,

$$(1-u_{n+1})(1-u_{n+2})\dots(1-u_{n+r})$$
$$> 1-(u_{n+1}+u_{n+2}+\dots+u_{n+r}) > 1-\epsilon,$$

and therefore $|(1-u_{n+1})(1-u_{n+2})\dots(1-u_{n+r})-1| < \epsilon$, and thus the condition obtained in Art. 279 for the convergence of the infinite product $\Pi(1-u)$ is satisfied.

Also

$$(1+u_{n+1})(1+u_{n+2})\dots(1+u_{n+r})$$
$$< \frac{1}{(1-u_{n+1})(1-u_{n+2})\dots(1-u_{n+r})} < \frac{1}{1-\epsilon};$$

and thus $|(1+u_{n+1})(1+u_{n+2})\dots(1+u_{n+r})-1| < \frac{\epsilon}{1-\epsilon}$. If η be arbitrarily assigned, we can determine ϵ so that $\epsilon/(1-\epsilon) < \eta$, and thus n can be so determined that

$$|(1+u_{n+1})(1+u_{n+2})\dots(1+u_{n+r})-1| < \eta,$$

for $r = 1, 2, 3, \dots$. Hence the product $\Pi(1+u)$ is convergent. It is clear that the condition that $u_1, u_2, \dots u_n, \dots$ should all be less than 1 can be replaced by the wider condition that all except a finite set of these numbers are less than 1. For we can remove a finite set of factors in $\Pi(1+u)$ or in $\Pi(1-u)$ without affecting its convergence.

281. Next let us consider the infinite product

$$(1+u_1)(1+u_2)\dots(1+u_n)\dots,$$

where $u_1, u_2, \dots u_n, \dots$ are complex numbers. We shall shew that if the series of moduli of $u_1, u_2, \dots u_n, \dots$, *i.e.* the series,

$$|u_1| + |u_2| + \dots + |u_n| + \dots,$$

is convergent, then the infinite product is also convergent. In this case the infinite product is said to be absolutely convergent.

We see that

$$|(1+u_n)(1+u_{n+1})\dots(1+u_{n+r})-1|$$
$$\leqq (1+|u_n|)(1+|u_{n+1}|)\dots(1+|u_{n+r}|)-1,$$

since the modulus of the sum of any set of numbers cannot exceed the sum of their moduli. Now if the series $\Sigma|u|$ is convergent, the infinite product $\Pi(1+|u|)$ is convergent, in accordance with what has been shewn in Art. 280; it follows that, corresponding to any assigned ϵ, n can be so determined that

$$(1+|u_n|)(1+|u_{n+1}|)\dots(1+|u_{n+r}|)-1 < \epsilon,$$

for $r = 1, 2, 3, \dots$. It follows that

$$|(1+u_n)(1+u_{n+1})\dots(1+u_{n+r})-1| < \epsilon,$$

for all positive integral values of r, and therefore the product $\Pi(1+u)$ is convergent. It may happen that $\Pi(1+u)$ is convergent whilst the series $\Sigma|u|$ is divergent; in this case $\Pi(1+u)$ is said to converge non-absolutely, or to be semi-convergent.

It follows from the above theorem that the infinite product

$$(1+a_1z)(1+a_2z)\dots(1+a_nz)\dots$$

is convergent if $\quad |a_1| + |a_2| + \dots |a_n| + \dots$

is a convergent series.

Let $b_1, b_2, b_3, \dots b_n, \dots$ be a sequence of real numbers all of the same sign, and let $\underset{n=\infty}{L} b_n = 0$, but suppose the series

$$b_1 + b_2 + \dots + b_n + \dots$$

to be divergent. It will be shewn that the infinite product $\Pi(1+ib_n)$ is not convergent. To prove this we see that $1+ib_n = (1+b_n^2)^{\frac{1}{2}} e^{\pm i\phi_n}$, when $\tan\phi_n = |b_n|$, and the upper or lower sign in $\pm i\phi_n$ is taken according as b_n is positive or negative. If η be an arbitrarily chosen positive number less than unity, we have $\phi_n > (1-\eta)\tan\phi_n$, for all sufficiently large values of n; and therefore $\Sigma\phi_n$ cannot converge. It follows that $\Pi(1+ib_n)$ cannot converge, although $\Pi(1+b_n^2)^{\frac{1}{2}}$ will converge in case the series Σb_n^2 is convergent. It is clearly sufficient for the validity of the theorem that all the numbers b_n, with the exception of a finite set, should be of the same sign.

If z be a complex number $x+iy$, and the numbers $a_1, a_2, \dots a_n, \dots$ be all positive and such that Σa_n is divergent, the product $\Pi(1+a_n z)$ is certainly divergent if the real part of z is positive. For the product of the moduli of the terms $1+a_n z$ is greater than $\Pi(1+a_n x)$, and this is divergent when x is positive.

The product $\left(1+\frac{x}{1^p}\right)\left(1+\frac{x}{2^p}\right)\dots\left(1+\frac{x}{n^p}\right)\dots$, when x is a real number, does not converge in case $p \leqq 1$, but converges if $p>1$. For $\Sigma\frac{1}{n^p}$ is divergent when $p \leqq 1$, and is convergent if $p>1$.

The product $\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right)\dots$ is certainly divergent if the real part of z is positive, and it does not converge if the real part of z is zero. When the real part of z is negative the product converges to zero, and is therefore considered as non-convergent. For $\log_e\left(1+\frac{z}{n}\right) = \frac{z}{n} - \frac{z^2}{2n^2}(1+\eta_n)$, where $|\eta_n|$ is less than a fixed number for all sufficiently large values of n; the real part of $\Sigma\log_e\left(1+\frac{z}{n}\right)$ consequently diverges to $-\infty$ when the real part of z is negative, whence the result follows. This depends on the facts that $\Sigma\frac{1}{n}$ is divergent and $\Sigma\frac{1}{n^2}$ convergent.

Expressions for the sine and cosine as infinite products.

282. We shall now find expressions for $\sin x$, $\cos x$ as infinite products involving the circular measure x; we first suppose x to be real and positive.

We have

$$\sin x = 2\sin\frac{x}{2}\sin\frac{x+\pi}{2}$$

$$= 2^2\sin\frac{x}{4}\sin\frac{x+\pi}{4}\sin\frac{x+2\pi}{4}\sin\frac{x+3\pi}{4},$$

and continuing this process, we obtain

$$\sin x = 2^{n-1}\sin\frac{x}{n}\sin\frac{x+\pi}{n}\sin\frac{x+2\pi}{n}\dots\sin\frac{x+(n-1)\pi}{n},$$

where n is any positive integral power of 2; hence

$$\sin x = 2^{n-1}\sin\frac{x}{n}\cos\frac{x}{n}\left(\sin^2\frac{\pi}{n}-\sin^2\frac{x}{n}\right)$$

$$\left(\sin^2\frac{2\pi}{n}-\sin^2\frac{x}{n}\right)\dots\left(\sin^2\frac{\overline{n-2}\,\pi}{2n}-\sin^2\frac{x}{n}\right);$$

since $\underset{x=0}{L}\sin x\operatorname{cosec}\frac{x}{n}=n$, we have

$$n = 2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\dots\sin^2\frac{\overline{n-2}\,\pi}{2n};$$

hence, by division we find

$$\frac{\sin x}{n\sin\frac{x}{n}\cos\frac{x}{n}} = \left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{2\pi}{n}}\right)\dots\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\overline{n-2}\,\pi}{2n}}\right).$$

This is the particular case of the theorem (19), of Art. 87, when n is a power of 2. We might, of course, assume the general theorem.

Let $\frac{1}{2}(n-2)=r$, then if m be any number less than r, we have

$$\sin x = n\sin\frac{x}{n}\cos\frac{x}{n}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{2\pi}{n}}\right)\dots\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{m\pi}{n}}\right)R,$$

where $$R = \left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\overline{m+1}\,\pi}{n}}\right)\dots\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right).$$

Now, n being taken greater than $2x/\pi$, m may be so chosen that $x < (m+1)\pi$, then R is positive and less than unity; also, as in Art. 226, R is greater than

$$1 - \sin^2\frac{x}{n}\left\{\operatorname{cosec}^2\frac{\overline{m+1}\pi}{n} + \ldots + \operatorname{cosec}^2\frac{r\pi}{n}\right\}.$$

Now we have shewn in Art. 96, Ex. (1), that if $\theta < \frac{1}{2}\pi$,

then $$\frac{\sin\theta}{\theta} > \frac{\sin\frac{1}{2}\pi}{\frac{1}{2}\pi},$$

hence, if $$p < \frac{n}{2},\ \operatorname{cosec}^2\frac{p\pi}{n} < \frac{n^2}{4p^2};\ \text{also } \sin^2\frac{x}{n} < \frac{x^2}{n^2},$$

hence $$R > 1 - \frac{x^2}{4}\left\{\frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \ldots + \frac{1}{r^2}\right\},$$

$$> 1 - \frac{x^2}{4}\left\{\frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \ldots + \frac{1}{(r-1)r}\right\},$$

$$> 1 - \frac{x^2}{4}\left(\frac{1}{m} - \frac{1}{r}\right) > 1 - \frac{x^2}{4m}.$$

Since R is between 1 and $1 - \frac{x^2}{4m}$, we may put $R = 1 - \frac{\theta x^2}{4m}$, where θ is between 0 and 1; we have then

$$\sin x = n\sin\frac{x}{n}\cos\frac{x}{n}\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{2\pi}{n}}\right)\ldots$$
$$\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{m\pi}{n}}\right)\left(1 - \frac{\theta x^2}{4m}\right),$$

where m is any number less than $\frac{1}{2}n$, such that $x < (m+1)\pi$.

Now let n become indefinitely great, m remaining fixed, we have then, since each sine in the product may be replaced by the corresponding circular measure, and since $\cos\frac{x}{n}$ has the limit unity,

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\ldots\left(1 - \frac{x^2}{m^2\pi^2}\right)\left(1 - \frac{\theta_1 x^2}{4m}\right),$$

where θ_1 is the limiting value of θ, when n is indefinitely increased, and is thus such that $0 \leqq \theta_1 \leqq 1$.

Now by increasing m sufficiently, we may make the factor $1 - \frac{\theta_1 x^2}{4m}$ as nearly equal to unity as we please, hence we have the expression

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\ldots\ldots\ldots(1),$$

for $\sin x$ as an infinite product[1]. The restriction that x should be positive may clearly be removed.

283. From the formula (17), in Art. 86, if n is even,

$$\cos x = \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{3\pi}{2n}}\right)\ldots\left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\overline{n-1}\,\pi}{2n}}\right),$$

we may shew that

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\ldots\left(1 - \frac{4x^2}{\overline{2m-1}|^2\pi^2}\right)\left(1 - \frac{\theta x^2}{2m}\right),$$

where m is any finite number such that $2x < (2m+1)\pi$, and θ is between 0 and 1; hence we obtain for $\cos x$ as an infinite product, the formula

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\ldots\ldots\ldots\ldots(2).$$

284. On account of the importance of the formulae (1) and (2), we shall give another proof, taken from Serret's Trigonometry. Taking the formulae

$$\sin x = n \sin\frac{x}{n}\cos\frac{x}{n} \prod_{r=1}^{r=\frac{1}{2}(n-2)} \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right),$$

$$\cos x = \prod_{r=1}^{r=\frac{1}{2}n} \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{(2r-1)\pi}{2n}}\right),$$

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[1] The investigation of this Article is due to Schlomilch, see his *Compendium der höheren Analysis*, Vol. I.

which hold for even values of n, we transform them by means of the formula $1-\dfrac{\sin^2\alpha}{\sin^2\beta}=\cos^2\alpha\left(1-\dfrac{\tan^2\alpha}{\tan^2\beta}\right)$, into the forms

$$\sin x = n\cos^n\frac{x}{n}\,.\,\tan\frac{x}{n}\prod_{r=1}^{r=\frac{1}{2}(n-2)}\left(1-\frac{\tan^2\dfrac{x}{n}}{\tan^2\dfrac{r\pi}{n}}\right),$$

$$\cos x=\cos^n\frac{x}{n}\,.\prod_{r=1}^{r=\frac{1}{2}n}\left(1-\frac{\tan^2\dfrac{x}{n}}{\tan^2\dfrac{(2r-1)\pi}{2n}}\right).$$

Now it has been shewn in Art. 96, Ex. (1), that as θ increases from 0 to $\frac{1}{2}\pi$, $\dfrac{\sin\theta}{\theta}$ diminishes, and $\dfrac{\tan\theta}{\theta}$ increases, hence

$$\left(1\sim\frac{\sin^2\alpha}{\sin^2\beta}\right)<\left(1\sim\frac{\alpha^2}{\beta^2}\right)<\left(1\sim\frac{\tan^2\alpha}{\tan^2\beta}\right),$$

where the absolute value of each expression is to be taken. Suppose n so large that $\pm x/n<\frac{1}{2}\pi$, then $\pm\sin\dfrac{x}{n}<\pm\dfrac{x}{n}<\pm\tan\dfrac{x}{n}$, and $\pm\cos\dfrac{x}{n}<1$, the signs being so taken that each expression has its arithmetical value; the two expressions for $\sin x$ shew that

$$\pm\sin x<\pm x\prod_{r=1}^{r=\frac{1}{2}(n-2)}\left(1-\frac{x^2}{r^2\pi^2}\right),$$

and $$\pm\sin x>\pm\cos^n\frac{x}{n}\,.\,x\prod_{r=1}^{r=\frac{1}{2}(n-2)}\left(1-\frac{x^2}{r^2\pi^2}\right),$$

and the two expressions for $\cos x$ shew that

$$\pm\cos x<\pm\prod_{r=1}^{r=\frac{1}{2}n}\left(1-\frac{4x^2}{(2r-1)^2\pi^2}\right),$$

and $$\pm\cos x>\pm\cos^n\frac{x}{n}\prod_{r=1}^{r=\frac{1}{2}n}\left(1-\frac{4x^2}{\overline{2r-1}|^2\pi^2}\right);$$

now we know that $\cos^n\dfrac{x}{n}=1-\epsilon_n$, where ϵ_n is a number which converges to zero as n is indefinitely increased; we have therefore

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\dots\left(1-\frac{x^2}{n^2\pi^2}\right)(1-\theta_n),$$

$$\cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\dots\left(1-\frac{4x^2}{\overline{2n-1}|^2\pi^2}\right)(1-\theta_n'),$$

where θ_n, θ_n' are numbers which converge to zero when n is indefinitely increased; we thus obtain the expressions (1) and (2).

If we had used the formulae

$$\sin x = n \sin\frac{x}{n} \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right),$$

$$\cos x = \cos\frac{x}{n} \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\sin^2 x}{\sin^2\frac{\overline{2r-1}\pi}{2n}}\right),$$

which hold for an odd value of n,

and the formulae

$$\sin x = \cos^n\frac{x}{n} \, . \tan\frac{x}{n} \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\tan^2\frac{x}{n}}{\tan^2\frac{r\pi}{n}}\right),$$

$$\cos x = \cos^n\frac{x}{n} \prod_{r=1}^{r=\frac{1}{2}(n-1)} \left(1 - \frac{\tan^2\frac{x}{n}}{\tan^2\frac{\overline{2r-1}\pi}{2n}}\right),$$

obtained from them, similar reasoning would have led to the same results.

285. We shall next consider the case of a complex variable $z = x + iy$; we find, as in Art. 282,

$$\sin z = n \sin\frac{z}{n}\cos\frac{z}{n}\left(1 - \frac{\sin^2\frac{z}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1 - \frac{\sin^2\frac{z}{n}}{\sin^2\frac{2\pi}{n}}\right)\cdots\left(1 - \frac{\sin^2\frac{z}{n}}{\sin^2\frac{m\pi}{n}}\right)R,$$

where $$R = \left(1 - \frac{\sin^2\frac{z}{n}}{\sin^2\frac{\overline{m+1}\pi}{n}}\right)\cdots\left(1 - \frac{\sin^2\frac{z}{n}}{\sin^2\frac{r\pi}{n}}\right),$$

where n is an even integer, and $r = \frac{1}{2}(n-2)$; we have to determine limits for the value of R. Let ρ denote the modulus of $\sin\frac{z}{n}$, then as in Art. 281, since the modulus of the sum of any numbers is less than the sum of their moduli, we see that the modulus of $(R-1)$ is less than

$$\left(1 + \frac{\rho^2}{\sin^2\frac{\overline{m+1}\pi}{n}}\right)\cdots\left(1 + \frac{\rho^2}{\sin^2\frac{r\pi}{n}}\right) - 1.$$

Now we know that $e^{A\rho^2} > 1 + A\rho^2$, if A is any positive number, hence the modulus of $R-1$ is less than

$$e^{\rho^2\left(\operatorname{cosec}^2\frac{\overline{m+1}\pi}{n}+\ldots+\operatorname{cosec}^2\frac{r\pi}{n}\right)}-1,$$

and this is less than

$$e^{\frac{1}{4}\rho^2n^2\left\{\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\ldots+\frac{1}{r^2}\right\}}-1,$$

or than
$$e^{\frac{1}{4}\rho^2n^2\left\{\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+1}-\frac{1}{m+2}+\ldots-\frac{1}{r}\right\}}-1,$$

therefore the modulus of $(R-1)$ is less than

$$e^{\frac{1}{4}\rho^2n^2\left(\frac{1}{m}-\frac{1}{r}\right)}-1, \text{ or than } e^{\frac{1}{4}\frac{\rho^2n^2}{m}}-1;$$

thus the modulus of $(R-1)$ lies between zero and $e^{\frac{1}{4}\frac{\rho^2n^2}{m}}-1$. Now

$$\rho^2 = \sin^2\frac{x}{n}\cosh^2\frac{y}{n}+\cos^2\frac{x}{n}\sinh^2\frac{y}{n}=\sin^2\frac{x}{n}+\sinh^2\frac{y}{n},$$

hence the limiting value of ρ^2n^2 is x^2+y^2, therefore the limit of the modulus of $(R-1)$, when n is increased indefinitely, lies between zero and $e^{\frac{x^2+y^2}{4m}}-1$; now $e^{\frac{x^2+y^2}{4m}}$ may be made as near unity as we please, by taking m large enough, thus $|R-1|$ may be made as small as we please, by taking m large enough. When n is indefinitely increased, each of the sines in the expression for $\sin z$ becomes ultimately equal to its argument, therefore

$$\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\left(1-\frac{z^2}{3^2\pi^2}\right)\ldots.$$

The formula

$$\cos z = \left(1-\frac{4z^2}{1^2\pi^2}\right)\left(1-\frac{4z^2}{3^2\pi^2}\right)\left(1-\frac{4z^2}{5^2\pi^2}\right)\ldots$$

may be proved in a similar manner.

286. We remark about the formulae (1) and (2), that they satisfy the condition of absolute convergency given in Art. 281, since the two series $\frac{x^2}{\pi^2}\sum\limits_1^\infty\frac{1}{n^2}$ and $\frac{4x^2}{\pi^2}\sum\limits_1^\infty\frac{1}{(2r-1)^2}$ are convergent.

Each quadratic factor in either product may be resolved into two factors linear in x, thus

$$\sin x = x\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{2\pi}\right)\ldots,$$

$$\cos x = \left(1+\frac{2x}{\pi}\right)\left(1-\frac{2x}{\pi}\right)\left(1+\frac{2x}{3\pi}\right)\left(1-\frac{2x}{3\pi}\right)\ldots,$$

which may be written in the forms

$$\sin x = x\prod_{-\infty}^{+\infty}\left(1+\frac{x}{r\pi}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\cos x = \prod_{-\infty}^{\infty}\left(1+\frac{2x}{\overline{2r-1}\pi}\right)\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

In these latter forms, the products are semi-convergent, since the products

$$\prod_{1}^{\infty}\left(1+\frac{x}{r\pi}\right),\ \prod_{1}^{\infty}\left(1-\frac{x}{r\pi}\right),\ \prod_{1}^{\infty}\left(1+\frac{2x}{\overline{2r-1}\pi}\right),\ \prod_{1}^{\infty}\left(1-\frac{2x}{\overline{2r-1}\pi}\right),$$

are divergent, the series $\sum_{1}^{\infty}\frac{1}{r}$, $\sum_{1}^{\infty}\frac{1}{2r-1}$ being divergent. A semi-convergent product has the property analogous to that of semi-convergent séries, that a derangement of the order of the factors affects the value of the product; we are entitled to consider the formulae (3) and (4), as correct, only when it is understood that an equal number of positive and of negative values of r are to be taken; thus (3) and (4) must be regarded as an abbreviation of the forms

$$\sin x = x L_{n=\infty}\prod_{-n}^{n}\left(1+\frac{x}{r\pi}\right),\quad \cos x = L_{n=\infty}\prod_{-n}^{n}\left(1+\frac{2x}{\overline{2r-1}\pi}\right).$$

287. It has been shewn by *Weierstrass*[1], that the divergent product

$$z\left(1+\frac{z}{\pi}\right)\left(1+\frac{z}{2\pi}\right)\left(1+\frac{z}{3\pi}\right)\ldots$$

may be made convergent, by multiplying each factor by an exponential factor; thus the product

$$z\left\{\left(1+\frac{z}{\pi}\right)e^{-\frac{z}{\pi}}\right\}\left\{\left(1+\frac{z}{2\pi}\right)e^{-\frac{z}{2\pi}}\right\}\left\{\left(1+\frac{z}{3\pi}\right)e^{-\frac{z}{3\pi}}\right\}\ldots$$

is absolutely convergent.

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[1] See the *Abhandlungen* of the Berlin Academy, for 1876.

We have, as has been shewn in Art. $230^{(1)}$,

$$e^{-\frac{z}{n\pi}} = 1 - \frac{z}{n\pi} + \frac{z^2}{2n^2\pi^2}(1 + u_n),$$

where $|u_n|$ converges to zero as n is indefinitely increased; therefore, if ϵ be an arbitrarily chosen positive number, $|u_n| < \epsilon$, for all values of n which exceed some fixed value dependent on ϵ. We have now

$$\left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}} = \left(1 + \frac{z}{n\pi}\right)\left\{1 - \frac{z}{n\pi} + \frac{z^2}{2n^2\pi^2}(1 + u_n)\right\}$$
$$= 1 - \frac{z^2}{2n^2\pi^2}(1 - u_n) + \frac{z^3}{2n^3\pi^3}(1 + u_n).$$

The series of which the general term is

$$\frac{z^2}{2n^2\pi^2}\left\{1 - u_n - \frac{z}{n\pi}(1 + u_n)\right\}$$

is absolutely convergent, since the series $\Sigma\frac{1}{n^2}$, $\Sigma\frac{1}{n^3}$ are convergent, and $|u_n| < \epsilon$, $|1 \pm u_n| < 1 + \epsilon$, for all sufficiently large values of n. Therefore, in accordance with the theorem proved in Art. 281, the infinite product of which the general term is

$$1 - \frac{z^2}{2n^2\pi^2}(1 - u_n) + \frac{z^3}{2n^3\pi^3}(1 + u_n),$$

or $\left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}}$, is absolutely convergent.

If $f(z)$ denote the limit of the absolutely convergent product $\prod\limits_1^\infty \left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}}$, and $f(-z)$ that of $\prod\limits_1^\infty \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}}$, we have

$$f(z) f(-z) = \frac{\sin z}{z}.$$

The above result may be employed to evaluate the limiting value of the expression

$$\phi(z) = \left(1 - \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\cdots\left(1 - \frac{z}{n\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 + \frac{z}{2\pi}\right)$$
$$\cdots\left(1 + \frac{z}{m\pi}\right)$$

when m and n are made indefinitely great, but so that their ratio has a definite finite limit.

If s_n denotes the series $1^{-1} + 2^{-1} + 3^{-1} + \ldots + n^{-1}$, we see that

$$\sin z = z\, L\phi(z) \,.\, e^{\frac{z}{\pi}(s_n - s_m)};$$

now it is well known that the limit, when n is infinite, of $s_n - \log_e n$ is a finite number 0·5772156..., called Euler's constant, hence the limiting value of $s_n - s_m$, when m and n are infinite, is that of $\log_e \frac{n}{m}$. We have therefore,

$$L\phi(z) = k^{z/\pi}\frac{\sin z}{z},$$

where $k = Lm/n$, and the value of $L\phi(z)$ is $\frac{\sin z}{z}$ only when m and n become infinite in a ratio of equality.

288. The formulae (2) or (4), for $\cos x$, may be deduced from (1) or (3), by means of the formula $\cos x = \sin 2x/2\sin x$.

We have

$$\frac{\sin 2x}{2\sin x} = 2x\prod_{-\infty}^{\infty}\left(1+\frac{2x}{r\pi}\right)\bigg/ 2x\prod_{-\infty}^{\infty}\left(1+\frac{x}{r\pi}\right),$$

the factors in the numerator, for which r is even, cancel with those in the denominator, hence if we consider the product in the numerator to be the limit of $\prod_{-2n}^{2n}\left(1+\frac{2x}{r\pi}\right)$, and that in the denominator to be the limit of $\prod_{-n}^{n}\left(1+\frac{x}{r\pi}\right)$, when n is infinite, we see that $\cos x = \prod_{-\infty}^{\infty}\left(1+\frac{2x}{\overline{2s+1}\,\pi}\right)$ which agrees with (2) or (4). The condition of convergence of the products shews that taking $2n$ instead of n, in one of the products, does not affect the limiting value of that product when n is indefinitely increased.

289. We may deduce the product formula for $\sin x$ from that of $\cos x$, or vice versa, by means of the formulae $\sin x = \cos(\frac{1}{2}\pi - x)$, $\cos x = \sin(\frac{1}{2}\pi - x)$. From the formula (4) we have

$$\sin x = \prod_{-\infty}^{\infty}\left(1+\frac{\pi-2x}{\overline{2r-1}\,\pi}\right) = \prod_{-\infty}^{\infty}\left(\frac{2r\pi-2x}{\overline{2r-1}\,\pi}\right)$$

$$= \prod_{-\infty}^{\infty}\frac{2r}{2r-1}\cdot x\prod_{-\infty}^{\infty}\left(1-\frac{x}{r\pi}\right),$$

where the factor x corresponds to $r=0$; taking the limit of $\frac{\sin x}{x}$ for $x=0$, we see that we must have $\prod_{-\infty}^{\infty}\frac{2r}{2r-1} = 1$,

hence $$\sin x = x\prod_{-\infty}^{\infty}\left(1-\frac{x}{r\pi}\right).$$

290. The product formulae for $\sin x$ and $\cos x$ may be easily made to exhibit the property of periodicity which those functions possess.

Let
$$f(x) = x \prod_{-n}^{n} \left(1 + \frac{x}{r\pi}\right),$$
then
$$f(x+\pi) = (x+\pi)\left(1+\frac{x+\pi}{\pi}\right)\left(1+\frac{x+\pi}{2\pi}\right)\ldots$$
$$\left(1+\frac{x+\pi}{n\pi}\right)\left(1-\frac{x+\pi}{\pi}\right)\ldots\left(1-\frac{x+\pi}{n\pi}\right)$$
$$= -x\left(1+\frac{x}{\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\left(1+\frac{x}{\overline{n+1}\pi}\right)\left(1-\frac{x}{\pi}\right)\ldots$$
$$\left(1-\frac{x}{\overline{n-1}\pi}\right)\frac{n+1}{n}$$
$$= -\frac{x+(n+1)\pi}{n\pi - x} f(x);$$

now when n is indefinitely increased, we have $Lf(x+\pi) = -Lf(x)$, which is the equation $\sin(x+\pi) = -\sin x$; the formula (4) may be made, in a similar manner, to exhibit the property
$$\cos(x+\pi) = -\cos x.$$

The function $\sin x$ vanishes when $x=0$, $\pm\pi$, $\pm 2\pi\ldots$, and these values correspond to the factors x, $1\pm\frac{x}{\pi}$, $1\pm\frac{x}{2\pi}\ldots$ in the formula (3); also it has been proved in Art. 235, that $\sin x$ does not vanish for any imaginary value of x, thus if it be assumed that $\sin x$ can be expressed in the form of an infinite product $A\frac{(x-a)(x-b)(x-c)\ldots}{bc\ldots}$, the values of $a, b, c\ldots$ must be 0, π, $-\pi$, 2π, $-2\pi\ldots$. The value of A is then determined by putting $x=0$, and using the theorem $L\frac{\sin x}{x}=1$, we obtain the formula (1) or (3). This is of course worthless as a proof of the formula, since we have no right to assume without proof that $\sin x$ is capable of expression in the required form.

291. It is important to notice the forms which the formulae (1) and (2) take in the case of an imaginary argument iy; we obtain in that case, the expressions for $\sinh y$, $\cosh y$ as infinite products
$$\sinh y = y\left(1+\frac{y^2}{\pi^2}\right)\left(1+\frac{y^2}{2^2\pi^2}\right)\left(1+\frac{y^2}{3^2\pi^2}\right)\ldots\ldots\ldots(5),$$
$$\cosh y = \left(1+\frac{4y^2}{\pi^2}\right)\left(1+\frac{4y^2}{3^2\pi^2}\right)\left(1+\frac{4y^2}{5^2\pi^2}\right)\ldots\ldots\ldots(6).$$

The formulae (1), (2), (5), (6) were first obtained by Euler, by means of the identity

$$z^{2m}-1=m(z^2-1)\prod_{n=1}^{n=m-1}\left\{\frac{1-2z\cos\frac{n\pi}{m}+z^2}{2-2\cos\frac{n\pi}{m}}\right\};$$

putting $z=1+\frac{x}{m}$, it becomes

$$\left(1+\frac{x}{m}\right)^m-\left(1+\frac{x}{m}\right)^{-m}=\frac{2x+\frac{x^2}{m}}{1+\frac{x}{m}}\prod_{n=1}^{n=m-1}\left\{1+\frac{x^2}{\left(1+\frac{x}{m}\right)\left(2m\sin\frac{n\pi}{2m}\right)^2}\right\};$$

if m be now made to increase indefinitely, this becomes

$$\tfrac{1}{2}(e^x-e^{-x})=x\prod_{n=1}^{n=\infty}\left(1+\frac{x^2}{n^2\pi^2}\right)$$

which is the formula (5). This evaluation of the limit requires an exact investigation, as in Art. 285.

The formula (1) was deduced by changing x into ix. The formulae (2), (6) were obtained in a similar manner, from the expression for $z^{2m}+1$ in factors.

EXAMPLES.

292. (1) *Investigate Wallis' expression for π.*

In the expression for $\sin x$ in factors, put $x=\frac{1}{2}\pi$, we have then the approximate formula

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\dots\dots\left(1-\frac{1}{\overline{2n|}^2}\right),$$

where n is large; this may be written

$$\sqrt{\tfrac{1}{2}\pi(2n+1)}=\frac{2.4.6\dots2n}{1.3.5\dots(2n-1)},$$

which is Wallis' formula.

(2) *Factorise* $\cosh y-\cos a$, $\cos x-\cos a$.

We have $\cosh y-\cos a=2\sin\frac{1}{2}(a+iy)\sin\frac{1}{2}(a-iy)$

$$=\tfrac{1}{2}(a^2+y^2)\prod_1^\infty\left\{1-\frac{(a+iy)^2}{4n^2\pi^2}\right\}\left\{1-\frac{(a-iy)^2}{4n^2\pi^2}\right\};$$

putting $y=0$, $\quad 1-\cos a=\frac{1}{2}a^2\prod_1^\infty\left(1-\frac{a^2}{4n^2\pi^2}\right)^2,$

hence

$$\frac{\cosh y-\cos a}{1-\cos a}$$

$$=\left(1+\frac{y^2}{a^2}\right)\prod_1^\infty\left(1+\frac{iy}{2n\pi+a}\right)\left(1-\frac{iy}{2n\pi-a}\right)\left(1-\frac{iy}{2n\pi+a}\right)\left(1+\frac{iy}{2n\pi-a}\right),$$

therefore

$$\cosh y-\cos a=2\sin^2\tfrac{1}{2}a.\left(1+\frac{y^2}{a^2}\right)\prod_1^\infty\left\{1+\frac{y^2}{(2n\pi+a)^2}\right\}\left\{1+\frac{y^2}{(2n\pi-a)^2}\right\}.$$

Writing ix for y, we have

$$\cos x - \cos a = 2\sin^2 \tfrac{1}{2}a \,.\left(1 - \frac{x^2}{a^2}\right) \prod_1^\infty \left\{1 - \frac{x^2}{(2n\pi + a)^2}\right\} \left\{1 - \frac{x^2}{(2n\pi - a)^2}\right\}.$$

(3) *Prove that*

$$tan^{-1}\frac{1}{\pi^2} + tan^{-1}\frac{1}{4\pi^2} + tan^{-1}\frac{1}{9\pi^2} + tan^{-1}\frac{1}{16\pi^2} + \ldots$$
$$= \tfrac{1}{4}\pi - tan^{-1}\left(tanh\frac{1}{\sqrt{2}} \,.\, cot\frac{1}{\sqrt{2}}\right).$$

We have $\sin(x+iy) = (x+iy)\prod_1^\infty \left\{1 - \frac{(x+iy)^2}{n^2\pi^2}\right\}$; taking logarithms, this becomes

$$\log(\sin x \cosh y + i\cos x \sinh y) = \log(x+iy) + \sum_1^\infty \log\left\{1 - \frac{x^2 - y^2}{n^2\pi^2} - i\,.\,\frac{2xy}{n^2\pi^2}\right\};$$

equating the imaginary parts on both sides of the equation, we have

$$\tan^{-1}(\tanh y \cot x) = \tan^{-1}\frac{y}{x} - \sum_1^\infty \tan^{-1}\frac{2xy}{n^2\pi^2 - x^2 + y^2};$$

let $$x = y = 1/\sqrt{2},$$

we have then

$$\sum_1^\infty \tan^{-1}\frac{1}{n^2\pi^2} = \tfrac{1}{4}\pi - \tan^{-1}\left(\tanh\frac{1}{\sqrt{2}} \,.\, \cot\frac{1}{\sqrt{2}}\right).$$

Representation of the exponential function by an infinite product.

292(1). A representation of the exponential function e^z, in the case in which $|z| < 1$, has been given by Mathews[1].

Let us assume that z is the limiting sum of a convergent series $\sum\limits_{n=1} k_n \log_e(1 + z^n)$. We find then that $k_1 = 1$, and

$$k_n + \Sigma \frac{(-1)^{\delta'-1}}{\delta'} k_\delta = 0,$$

for $n > 1$, where δ is any proper integral factor of n, and $\delta' = n/\delta$, each such value of δ giving one term. From this it follows that

$$nk_n = \Sigma(-1)^{\delta'}\delta k_\delta = \Sigma(-1)^{n/\delta}\delta k_\delta;$$

and the values of all the numbers k_n are to be determined from the set of equations of which this is the type. It can be shewn by induction that

(1) If $n = 2^m$, then $k_n = 1/2$.

(2) If n is the product $p_1 p_2 \ldots p_\mu$ of μ different odd primes, then $k_n = (-1)^\mu/n$.

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[1] *Proceedings of the Cambridge Philosophical Society*, Vol. XIV. p. 228.

(3) If $n = 2^m p_1 p_2 \ldots p_\mu$, then $k_n = (-1)^\mu 2^{m-1}/n$.

(4) If n has the square of an odd number as factor, then $k_n = 0$.

That, with the values of k_n so determined, the series

$$\Sigma k_n \log_e (1 + z^n)$$

converges when $|z| < 1$ is easily seen. The exponential function e^z is consequently represented, for all values of z such that $|z| < 1$, by the infinite product

$$\prod_1^\infty (1 + z^n)^{k_n} \equiv (1 + z)(1 + z^2)^{1/2}(1 + z^3)^{-1/6}(1 + z^4)^{1/2} \ldots ;$$

or, since $1 = (1 - z)^{1/2}(1 + z)^{1/2}(1 + z^2)^{1/2} \ldots$, we have by division

$$e^z = \left(\frac{1 + z}{1 - z}\right)^{1/2} \Pi \left(\frac{1 + z^p}{1 - z^p}\right)^{(-1) \quad /2p} . \quad |z| < 1,$$

where p is the product of μ unequal odd primes, and all values of p of this form are to be taken.

Series for the tangent, cotangent, secant, and cosecant.

293. Since $\sin z = z \prod_1^\infty \left(1 - \frac{z^2}{n^2\pi^2}\right)$, we have, when z is not a multiple of π,

$$\log_e \sin z = \log_e z + \sum_1^\infty \log_e \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Let h be a positive real number, changing z into $z + h$, and subtracting the two expressions, we have

$$\log_e \frac{\sin (z + h)}{\sin z}$$

$$= \log_e \left(1 + \frac{h}{z}\right) + \sum_{n=1}^\infty \left\{\log_e \left(1 + \frac{h}{z - n\pi}\right) + \log_e \left(1 + \frac{h}{z + n\pi}\right)\right\}.$$

Now, employing the theorem given in Art. 249 (1), we have

$$\log_e \left(1 + \frac{h}{z}\right) = \frac{h}{z} - \tfrac{1}{2}\frac{h^2}{z^2}(1 + v_0),$$

$$\log_e \left(1 + \frac{h}{z - n\pi}\right) = \frac{h}{z - n\pi} - \tfrac{1}{2}\frac{h^2}{(z - n\pi)^2}(1 + v_n),$$

$$\log_e \left(1 + \frac{h}{z + n\pi}\right) = \frac{h}{z + n\pi} - \tfrac{1}{2}\frac{h^2}{(z + n\pi)^2}(1 + w_n),$$

where $|v_0|, |v_n|, |w_n|$ all converge to zero when h is indefinitely diminished. Moreover, z having any fixed value which is not zero or a positive or negative integral multiple of π, for all sufficiently small values of h the numbers $|v_0|, |v_1|, |v_2| \dots$ and $|w_1|, |w_2| \dots$ are all less than an arbitrarily chosen positive number ϵ, since the moduli of $|z|, |z - n\pi|, |z + n\pi|$ are greater than some fixed number independent of n.

We have now

$$\frac{1}{h}\log_e \frac{\sin(z+h)}{\sin z}$$
$$= \left[\frac{1}{z} - \frac{1}{2}\frac{h}{z^2}(1+v_0)\right] + \sum_{n=1}^{\infty}\left[\frac{2z}{z^2 - n^2\pi^2} - \frac{1}{2}h\frac{1+v_n}{(z-n\pi)^2} - \frac{1}{2}h\frac{1+w_n}{(z+n\pi)^2}\right],$$

where the series on the right side converges when z is not a multiple of π.

Let us assume that z is such that $(r-1)\pi < |z| < r\pi$, where r is a positive integer; then if $z^2/r^2\pi^2 = \eta < 1$, we have $|z|^2/n^2\pi^2 \leqq \eta$, for all values of n which are $\geqq r$. We have now

$$\left|\frac{1}{n^2\pi^2 - z^2}\right| = \frac{1}{n^2\pi^2}\frac{1}{|1 - z^2/n^2\pi^2|} \leqq \frac{1}{n^2\pi^2}\frac{1}{1-\eta},$$

provided $n \geqq r$; it follows, since the series of which n^{-2} is the general term is convergent, that the series of which $\frac{1}{n^2\pi^2 - z^2}$ is the general term is absolutely convergent.

Since the two series of which the general terms are

$$\frac{2z}{z^2 - n^2\pi^2}, \quad \frac{2z}{z^2 - n^2\pi^2} - \frac{1}{2}h\frac{1+v_n}{(z-n\pi)^2} - \frac{1}{2}h\frac{1+w_n}{(z+n\pi)^2}$$

are both convergent, it follows that the series of which the general term is

$$\frac{1}{2}h\frac{1+v_n}{(z-n\pi)^2} + \frac{1}{2}h\frac{1+w_n}{(z+n\pi)^2}$$

is also convergent. If h be sufficiently small, the modulus of this general term is less than

$$\frac{1}{2}h(1+\epsilon)\left\{\frac{1}{|z-n\pi|^2} + \frac{1}{|z+n\pi|^2}\right\};$$

now $|z - n\pi| \geqq n\pi - |z| \geqq n\pi - (r+1)\pi$, hence

$$\frac{1}{|z-n\pi|^2} < \frac{1}{(n-r-1)^2\pi^2},$$

where $n > r + 1$, and it then follows that the series of which the general term is $\frac{1}{|z - n\pi|^2}$ is convergent. Similarly the series of which the general term is $\frac{1}{|z + n\pi|^2}$ is convergent.

We now see that the modulus of the sum of the series of which the general term is $\frac{1}{2}h\frac{1 + v_n}{(z - n\pi)^2} + \frac{1}{2}h\frac{1 + w_n}{(z + n\pi)^2}$ does not exceed a number $\frac{1}{2}h(1 + \epsilon)A(z)$, where $A(z)$ is a positive number dependent only on z; this modulus diminishes indefinitely as h is indefinitely diminished. It now follows that

$$\frac{1}{z} + \sum_{n=1} \frac{2z}{z^2 - n^2\pi^2} \text{ converges to } \underset{h=0}{L} \frac{1}{h} \log_e \frac{\sin(z + h)}{\sin z}.$$

Since $\frac{\sin(z+h)}{\sin z} = \cos h + \sin h \cot z = 1 + h \cot z (1 + \zeta)$,

where $|\zeta|$ converges to zero with h, we have

$$\frac{1}{h}\log_e \frac{\sin(z+h)}{\sin z} = \frac{1}{h}\log_e \{1 + h\cot z(1 + \zeta)\}$$
$$= \cot z(1 + \zeta)(1 + \zeta'),$$

where $|\zeta'|$ converges to zero with h; hence

$$\underset{h=0}{L} \frac{1}{h}\log_e \frac{\sin(z+h)}{\sin z} = \cot z.$$

It has now been shewn that when z is any real or complex number which is not an integral multiple of π, $\cot z$ is the sum of the convergent series

$$\frac{1}{z} + \frac{1}{z+\pi} + \frac{1}{z-\pi} + \frac{1}{z+2\pi} + \frac{1}{z-2\pi} + \dots \quad \dots\dots(7),$$

or

$$\frac{1}{z} + 2z \sum_{n=1} \frac{1}{z^2 - n^2\pi^2} \dots\dots\dots\dots\dots\dots\dots\dots(8).$$

In the form (7) the series is semi-convergent, and in the form (8) it is absolutely convergent, except for $z = 0, \pm\pi, \pm 2\pi, \dots$, for which values the series is divergent.

In order that the student may appreciate the necessity for the investigation in the text, we remark that if $f(z)$ be the sum of an infinite convergent series $u_1(z) + u_2(z) + \dots + u_n(z) + \dots$, we are not entitled to assume that

$$\underset{h=0}{L} \frac{f(z+h) - f(z)}{h} = \sum_1^\infty \underset{h=0}{L} \frac{u_r(z+h) - u_r(z)}{h}.$$

Suppose $R_m(z)$ is the remainder of the series after m terms, then

$$f(z)=u_1(z)+u_2(z)+\ldots+u_m(z)+R_m(z),$$

$$f(z+h)=u_1(z+h)+u_2(z+h)+\ldots+u_m(z+h)+R_m(z+h);$$

hence

$$\underset{h=0}{L}\frac{f(z+h)-f(z)}{h}=\sum_1^m \underset{h=0}{L}\frac{u_r(z+h)-u_r(z)}{h}+\underset{h=0}{L}\frac{R_m(z+h)-R_m(z)}{h};$$

now since the given series is convergent, $R_m(z)$, $R_m(z+h)$ become indefinitely small when m is indefinitely increased; it does not however necessarily follow that $\underset{h=0}{L}\frac{R_m(z+h)-R_m(z)}{h}$ does the same, and it is only when it does that we are entitled to employ the derived series to represent the derived function of $f(z)$. If for example $R_m(z)$ were of the form $\frac{A}{m}\sin mz$, we should find

$$L\frac{R_m(z+h)-R_m(z)}{h}=A\cos mz,$$

which does not converge to zero when m is indefinitely increased, but oscillates between the values $\pm A$.

294. From the expression

$$\cos z=\left(1-\frac{4z^2}{\pi^2}\right)\left(1-\frac{4z^2}{3^2\pi^2}\right)\left(1-\frac{4z^2}{5^2\pi^2}\right)\ldots$$

we obtain, by a method similar to that of the last Article, the infinite series

$$-\tan z=\frac{1}{z+\frac{1}{2}\pi}+\frac{1}{z-\frac{1}{2}\pi}+\frac{1}{z+\frac{3}{2}\pi}+\frac{1}{z-\frac{3}{2}\pi}+\ldots$$
$$+\frac{1}{z+\frac{1}{2}(2m-1)\pi}+\frac{1}{z-\frac{1}{2}(2m-1)\pi}+\ldots\ldots(9),$$

or

$$\tan z=8z\sum_1^\infty\frac{1}{(2m-1)^2\pi^2-4z^2}\quad\ldots\ldots\ldots\ldots(10);$$

the series (9) is semi-convergent, but (10) is absolutely convergent for all values of z except $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$....

295. We may find a series for $\operatorname{cosec} z$ by means of either of the formulae $\operatorname{cosec} z=\cot\frac{1}{2}z-\cot z$, $\operatorname{cosec} z=\frac{1}{2}\cot\frac{1}{2}z+\frac{1}{2}\tan\frac{1}{2}z$: using the first of these formulae, we find on substituting the series for the cotangents

$$\operatorname{cosec} z=\left[\frac{2}{z}+\frac{2}{z+2\pi}+\frac{2}{z-2\pi}+\frac{2}{z+4\pi}+\frac{2}{z-4\pi}+\ldots\right]$$
$$-\left[\frac{1}{z}+\frac{1}{z+\pi}+\frac{1}{z-\pi}+\frac{1}{z+2\pi}+\frac{1}{z-2\pi}+\frac{1}{z+3\pi}+\frac{1}{z-3\pi}+\ldots\right];$$

hence cosec z

$$=\frac{1}{z}-\frac{1}{z+\pi}-\frac{1}{z-\pi}+\frac{1}{z+2\pi}+\frac{1}{z-2\pi}-\frac{1}{z+3\pi}-\frac{1}{z-3\pi}+\ldots \quad (11),$$

or

$$\operatorname{cosec} z=\frac{1}{z}+\sum_{1}^{\infty}\frac{(-1)^r\,2z}{(z^2-r^2\pi^2)} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

In the formula (11), change z into $z+\frac{1}{2}\pi$; we have then

$$\sec z=\left(\frac{1}{z+\frac{1}{2}\pi}-\frac{1}{z-\frac{1}{2}\pi}\right)-\left(\frac{1}{z+\frac{3}{2}\pi}-\frac{1}{z-\frac{3}{2}\pi}\right)+\ldots \quad (13),$$

or

$$\sec z=4\Sigma\,\frac{(-1)^{r-1}(2r-1)\,\pi}{(2r-1)^2\,\pi^2-4z^2}\ldots\ldots\ldots\ldots\ldots(14);$$

this series, when r is large, has its general term approaching the value $\frac{(-1)^{r-1}}{2r-1}$, therefore the series is only semi-convergent.

The cotangent and tangent series may also be obtained as follows:

Using the expressions for $\sin(z+h)$ and $\sin z$ as infinite products, we find by division

$$\frac{\sin(z+h)}{\sin z}=\left(1+\frac{h}{z}\right)\left(\frac{\pi^2-z^2-h^2-2hz}{\pi^2-z^2}\right)\left(\frac{2^2\pi^2-z^2-h^2-2hz}{2^2\,\pi^2-z^2}\right)\ldots;$$

if we assume that the product on the right-hand side can be expanded in powers of h, by multiplication, and put the left-hand side in the form $\cos h+\sin h\cot z$, then expand in powers of h, and equate the coefficients of h on both sides of the equation, we find

$$\cot z=\frac{1}{z}+\frac{2z}{z^2-\pi^2}+\frac{2z}{z^2-2^2\pi^2}+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The justification for our assumption that the infinite product may be arranged in a series of ascending powers of h, the coefficients of which are the infinite series obtained by ordinary multiplication, would require an investigation of the conditions that such a process gives a correct result; to do this would however require certain general theorems for which we have no space. The tangent series may be obtained in a similar manner from the infinite product

$$\frac{\cos(z+h)}{\cos z}=\left(\frac{\pi^2-4z^2-4h^2-8hz}{\pi^2-4z^2}\right)\left(\frac{3^2\pi^2-4z^2-4h^2-8hz}{3^2\pi^2-4z^2}\right)\ldots.$$

If the cotangent of z is expressed in the form

$$\Pi\left(1-\frac{4z^2}{\overline{2m-1}|^2\pi^2}\right)\Big/ z\Pi\left(1-\frac{z^2}{m^2\pi^2}\right)$$

and this expression be transformed into partial fractions, the denominators of which are the factors in $z\Pi\left(1-\frac{z^2}{m^2\pi^2}\right)$, we should obtain the series (8); a similar remark applies to $\tan z$, $\sec z$, $\operatorname{cosec} z$. The series have been obtained[1] by Glaisher, directly, by carrying out this transformation.

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[1] See *Quarterly Journal*, Vol. XVII.

Expansion of the tangent, cotangent, secant and cosecant in powers of the argument.

296. We have shewn in Art. 293 that

$$\cot z = \frac{1}{z} - \sum_1^m \frac{2z}{r^2\pi^2 - z^2} + R_m,$$

where $|R_m|$ is a number which may be made as small as we please by taking m large enough. Now if the modulus of z is less than $r\pi$, we have

$$\frac{1}{r^2\pi^2 - z^2} = \frac{1}{r^2\pi^2}\left(1 + \frac{z^2}{r^2\pi^2} + \frac{z^4}{r^4\pi^4} + \dots + \frac{z^{2s}}{r^{2s}\pi^{2s}} + \dots\right);$$

hence if we suppose that the modulus of z is less than π, we may expand each of the fractions $1/(r^2\pi^2 - z^2)$ in this manner, and we have, arranging the result in powers of z, as we are entitled to do since each of the series is absolutely convergent,

$$\cot z = \frac{1}{z} - \frac{2z}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{m^2}\right) - \frac{2z^3}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \dots + \frac{1}{m^4}\right) - \dots$$
$$- \frac{2z^{2n-1}}{\pi^{2n}}\left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \dots + \frac{1}{m^{2n}}\right) - \dots + R_m;$$

let S_{2n} denote the sum of the convergent series

$$\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \dots + \frac{1}{m^{2n}} + \dots,$$

then $S_{2n} = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \dots + \frac{1}{m^{2n}} + \epsilon_{2n}$, where ϵ_{2n} is a number which may be made as small as we please, by making m large enough; we have then

$$\cot z = \frac{1}{z} - \frac{2z}{\pi^2}S_2 - \frac{2z^3}{\pi^4}S_4 - \dots - \frac{2z^{2n-1}}{\pi^{2n}}S_{2n} - \dots$$
$$+ R_m + \frac{2z}{\pi^2}\epsilon_2 + \frac{2z^3}{\pi^4}\epsilon_4 + \dots + \frac{2z^{2n-1}}{\pi^{2n}}\epsilon_{2n} + \dots.$$

We see that $\epsilon_2 > \epsilon_4 > \epsilon_6 \dots$, hence the modulus of

$$\frac{2z}{\pi^2}\epsilon_2 + \frac{2z^3}{\pi^4}\epsilon_4 + \dots$$

is less than ϵ_2 multiplied by the sum of $\frac{2|z|}{\pi^2} + \frac{2|z^3|^2}{\pi^4} + \dots$ which is a convergent series, since mod. $z < \pi$, therefore the modulus of

$\Sigma \dfrac{2z^{2n-1}}{\pi^{2n}} e_{2n}$ may be made as small as we please, by making m large enough. We have therefore the infinite series for cot z,

$$\cot z = \frac{1}{z} - \frac{2z}{\pi^2} S_2 - \frac{2z^3}{\pi^4} S_4 - \frac{2z^5}{\pi^6} S_6 - \dots \quad \dots\dots(15),$$

which holds for all values of z such that mod. $z < \pi$, and in particular for all real values of z between $\pm\pi$.

From the theorem

$$\tan z = 8 \sum_1^m \frac{z}{(2r-1)^2\pi^2 - 4z^2} + R_m',$$

we may obtain, in a similar manner, the series for tan z in ascending powers of z. This series may however be deduced from (15), by means of the identity $\tan z = \cot z - 2\cot 2z$; we find

$$\tan z = \frac{2(2^2-1)z}{\pi^2} S_2 + \frac{2(2^4-1)z^3}{\pi^4} S_4 + \frac{2(2^6-1)z^5}{\pi^6} S_6 + \dots(16),$$

which holds if the modulus of z is less than $\frac{1}{2}\pi$, and in particular for real values of z between $\pm\frac{1}{2}\pi$.

Substituting for $\cot\frac{1}{2}z$, $\cot z$ their values from (15), in the formula $\operatorname{cosec} z = \cot\frac{1}{2}z - \cot z$, we have

$$\operatorname{cosec} z = \frac{1}{z} + (2-1)\frac{z}{\pi^2} S_2 + \frac{2^3-1}{2^2} \cdot \frac{z^3}{\pi^4} S_4 + \frac{2^5-1}{2^4} \cdot \frac{z^5}{\pi^6} S_6 + \dots(17),$$

which holds if mod. $z < \pi$.

297. To obtain a formula for sec z, in powers of z, we use the formula

$$\sec z = 4\pi\left(\frac{1}{\pi^2 - 4z^2} - \frac{3}{3^2\pi^2 - 4z^2} + \frac{5}{5^2\pi^2 - 4z^2} - \dots + \frac{(-1)^{m-1}(2m-1)}{(2m-1)^2\pi^2 - 4z^2}\right) + R_m'';$$

supposing the modulus of z to be less than $\frac{1}{2}\pi$; we have on expanding each fraction

$$\sec z = \frac{2^2}{\pi}\left\{\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{m-1}}{2m-1}\right\} + \frac{2^4}{\pi^3} z^2 \left\{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots + \frac{(-1)^{m-1}}{(2m-1)^3}\right\} + \dots + \frac{2^{2n+2}}{\pi^{2n+1}} z^{2n} \left\{\frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \dots + \frac{(-1)^{m-1}}{(2m-1)^{2n+1}}\right\} + \dots + R_m''.$$

Now let Σ_{2n+1} denote the sum to infinity of the infinite series

$$\frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots,$$

and let the remainder after the first m terms be ϵ_{2n+1}, then we have

$$\sec z = \frac{2^2}{\pi}\Sigma_1 + \frac{2^4}{\pi^3}z^2\Sigma_3 + \dots + \frac{2^{2n+2}}{\pi^{2n+1}}z^n\Sigma_{2n+1} + \dots$$
$$+ R_m'' + \frac{2^2}{\pi}\epsilon_1 + \frac{2^4}{\pi^3}z^2\epsilon_3 + \dots;$$

let ϵ' be the greatest of the numbers $\epsilon_1, \epsilon_3, \dots$, then the modulus of $\frac{2^2}{\pi}\epsilon_1 + \frac{2^4}{\pi^3}z^2\epsilon_3 + \dots$ is less than ϵ' times the sum of

$$\frac{2^2}{\pi} + \frac{2^4}{\pi^3}|z|^2 + \frac{2^6}{\pi^5}|z|^4 + \dots,$$

which last series is convergent when the modulus of z is less than $\frac{1}{2}\pi$.

We have thus shewn that the remainder of the series we have obtained for $\sec z$ is a number of which the modulus diminishes indefinitely as m increases, hence we have for $\sec z$ the infinite series

$$\sec z = \frac{2^2}{\pi}\Sigma_1 + \frac{2^4}{\pi^3}z^2\Sigma_3 + \frac{2^6}{\pi^5}z^4\Sigma_5 + \dots \dots\dots\dots (18),$$

which holds if $\text{mod.}\, z < \frac{1}{2}\pi$.

298. It is a well-known theorem in Algebra, that the function $z/(e^z - 1)$, where e^z has its principal value, can be expanded in a series of the form

$$1 - \frac{1}{2}z + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \dots + (-1)^{n-1}\frac{B_n}{(2n)!}z^{2n} + \dots,$$

where $B_1, B_2, \dots B_n, \dots$ are certain numbers called *Bernouilli's numbers*, and that this expansion holds for all values of z for which the series is convergent.

If we multiply by $e^z - 1$ we have

$$z = \left\{z + \frac{z^2}{2!} + \dots + \frac{z^{2n}}{(2n)!} + \dots\right\}\left\{1 - \frac{1}{2}z + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \dots\right.$$
$$\left. + (-1)^{n-1}\frac{B_n}{(2n)!}z^{2n} + \dots\right\};$$

$|z|$ being taken so small that both the series on the right-hand side are absolutely convergent, we may multiply them together, and arrange the product in a series of powers of z; the resulting series will be absolutely convergent, hence equating the coefficients of the powers of z above the first, on the right-hand side, to zero, we have a series of equations

$$\frac{B_1}{2!}-\frac{1}{2}\frac{1}{2!}+\frac{1}{3!}=0,\quad -\frac{B_2}{4!}+\frac{1}{3!}\frac{B_1}{2!}-\frac{1}{4!}\frac{1}{2}+\frac{1}{5!}=0,$$

the general type of which is

$$\frac{B_n}{(2n)!}-\frac{1}{3!}\frac{B_{n-1}}{(2n-2)!}+\dots+\frac{(-1)^n}{(2n-1)!}\frac{B_1}{2!}-\frac{(-1)^{n-1}}{(2n)!}\frac{1}{2}+\frac{(-1)^n}{(2n+1)!}=0.$$

By means of these equations, the numbers $B_1, B_2, B_3, \dots$ may be calculated; we find

$$B_1=\tfrac{1}{6},\ B_2=\tfrac{1}{30},\ B_3=\tfrac{1}{42},\ B_4=\tfrac{1}{30},\ B_5=\tfrac{5}{66},\ B_6=\tfrac{691}{2730},\ B_7=\tfrac{7}{6},\ \&c.$$

299. The coefficients in the expansions of $\cot z$, $\tan z$, $\operatorname{cosec} z$, in powers of z, may be expressed in terms of Bernouilli's numbers.

We have $\quad \cot z = i\dfrac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}=i\left(1+\dfrac{2}{e^{2iz}-1}\right);$

hence, if mod. z is small enough,

$$\cot z=\frac{1}{z}-\frac{2^2B_1}{2!}z-\frac{2^4B_2}{4!}z^3-\dots-\frac{2^{2n}B_n}{(2n)!}z^{2n-1}-\dots \quad (19).$$

Also $\operatorname{cosec} z=\cot\tfrac{1}{2}z-\cot z$; hence we have the series

$$\operatorname{cosec} z=\frac{1}{z}+\frac{2(2-1)B_1}{2!}z+\frac{2(2^3-1)B_2}{4!}z^3+\dots$$
$$+\frac{2(2^{2n-1}-1)B_n}{(2n)!}z^{2n-1}+\dots \quad (20).$$

Again, since $\tan z=\cot z-2\cot 2z$, we have the series

$$\tan z=\frac{2^2(2^2-1)B_1}{2!}z+\frac{2^4(2^4-1)B_2}{4!}z^3+\dots$$
$$+\frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}z^{2n-1}+\dots \quad (21).$$

It has been shewn that the series (19) and (20) are convergent if mod. $z<\pi$, and that (21) is convergent if mod. $z<\tfrac{1}{2}\pi$.

The series in (19), (20), (21) must be identical with those in

(15), (16), (17), respectively; hence equating the coefficients in (19) to those in (15), we have

$$\frac{2}{\pi^2}S_2=\frac{2^2}{2\,!}B_1,\quad \frac{2}{\pi^4}S_4=\frac{2^4}{4\,!}B_2,\ldots\ \frac{2}{\pi^{2n}}S_{2n}=\frac{2^{2n}}{(2n)\,!}B_n;$$

hence using the values of B_1, B_2, ... in Art. 298, we have

$$S_2=\frac{\pi^2}{6},\quad S_4=\frac{\pi^4}{90},\quad S_6=\frac{\pi^6}{945},\quad S_8=\frac{\pi^8}{9450},\ldots\ S_{2n}=\frac{2^{2n-1}\pi^{2n}}{(2n)\,!}B_n,$$

thus S_{2n} may be calculated by means of the formulae which give B_n.

The series (19) and (21) give a ready means of calculating the tangent or cotangent of an angle, the first few terms of the series are

$$\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\ldots,$$

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots.$$

The calculation of $\tan\frac{m}{n}90°$, $\cot\frac{m}{n}90°$ may be carried out as follows:

$\tan(m/n\,90°)=$	$\cot(m/n\,90°)=$
$2mn/(n^2-m^2)\times\cdot 6366197723675$	$n/m\times\cdot 636619772367581$
$+m/n\times\cdot 2975567820597$	$-4mn/(4n^2-m^2)\times\cdot 3183098861837$
$+m^3/n^3\times\cdot 0186886502773$	$-m/n\times\cdot 2052888894145$
$+m^5/n^5\times\cdot 0018424752034$	$-m^3/n^3\times\cdot 0065510747882$
$+m^7/n^7\times\cdot 0001975800714$	$-m^5/n^5\times\cdot 0003450292554$
$+m^9/n^9\times\cdot 0000216977245$	$-m^7/n^7\times\cdot 0000202791060$
$+m^{11}/n^{11}\times\cdot 0000024011370$	$-m^9/n^9\times\cdot 0000012366527$
$+m^{13}/n^{13}\times\cdot 0000002664132$	$-m^{11}/n^{11}\times\cdot 0000000764959$
$+m^{15}/n^{15}\times\cdot 0000000295864$	$-m^{13}/n^{13}\times\cdot 0000000047597$
$+m^{17}/n^{17}\times\cdot 0000000032867$	$-m^{15}/n^{15}\times\cdot 0000000002969$
$+m^{19}/n^{19}\times\cdot 0000000003651$	$-m^{17}/n^{17}\times\cdot 0000000000185$
$+m^{21}/n^{21}\times\cdot 0000000000405$	$-m^{19}/n^{19}\times\cdot 0000000000011$
$+m^{23}/n^{23}\times\cdot 0000000000045$	
$+m^{25}/n^{25}\times\cdot 0000000000005$	

In these expressions, the terms $\frac{8z}{\pi^2-4z^2}$, $\frac{1}{z}-\frac{2z}{\pi^2-z^2}$, which occur in the formulae (10) and (8), are first calculated separately, the series being then more rapidly convergent.

These series are taken from Euler's *Analysis of the Infinite*; they are however given by him to twenty places of decimals.

Series for the logarithmic sine and cosine.

300. We have shewn in Art. 285 that

$$\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\dots\left(1-\frac{z^2}{m^2\pi^2}\right)(1-\theta_m),$$

$$\cos z = \left(1-\frac{4z^2}{\pi^2}\right)\left(1-\frac{4z^2}{3^2\pi^2}\right)\dots\left(1-\frac{4z^2}{\overline{2m-1}|^2\pi^2}\right)(1-\theta_m'),$$

where θ_m, θ_m' are numbers whose moduli may be made as small as we please by taking m large enough; taking logarithms, we have

$$\log\sin z = \log z + \log\left(1-\frac{z^2}{\pi^2}\right) + \log\left(1-\frac{z^2}{2^2\pi^2}\right) + \dots$$
$$+ \log\left(1-\frac{z^2}{m^2\pi^2}\right) + \log(1-\theta_m),$$

$$\log\cos z = \log\left(1-\frac{4z^2}{\pi^2}\right) + \log\left(1-\frac{4z^2}{3^2\pi^2}\right) + \dots$$
$$+ \log\left(1-\frac{4z^2}{\overline{2m-1}|^2\pi^2}\right) + \log(1-\theta_m');$$

expanding the logarithms, we have, assuming that $|z| < \pi$ in the first case and $< \frac{\pi}{2}$ in the second case, so that the logarithms may be expanded in absolutely convergent series of powers of z,

$$\log\frac{\sin z}{z} = -\sum_{n=0}^{n=\infty}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\dots+\frac{1}{m^{2n}}\right)\frac{z^{2n}}{n\pi^{2n}} + \log(1-\theta_m),$$

$$\log\cos z = -\sum_{n=0}^{n=\infty}\left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\dots+\frac{1}{\overline{2m-1}|^{2n}}\right)\frac{2^{2n}z^{2n}}{n\pi^{2n}} + \log(1-\theta_m').$$

Now

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots$$
$$= \left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\dots\right)+\frac{1}{2^{2n}}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right);$$

hence $$\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\dots = \frac{2^{2n}-1}{2^{2n}}S_{2n},$$

we have therefore

$$\log\frac{\sin z}{z} = -\Sigma\frac{z^{2n}}{n\pi^{2n}}S_{2n} + \Sigma\frac{z^{2n}}{n\pi^{2n}}\epsilon_{2n} + \log(1-\theta_m),$$

$$\log\cos z = -\Sigma\frac{2^{2n}-1}{n\pi^{2n}}z^{2n}S_{2n} + \Sigma\frac{2^{2n}z^{2n}}{n\pi^{2n}}\eta_{2n} + \log(1-\theta_m'),$$

where ϵ_{2n}, η_{2n} are the remainders after m terms in the two series

$$\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \ldots,\quad \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \ldots.$$

The modulus of $\Sigma\frac{z^{2n}}{n\pi^{2n}}\epsilon_{2n}$ is less than $\epsilon'\Sigma\frac{|z|^{2n}}{n\pi^{2n}}$, and that of $\Sigma\frac{2^{2n}z^{2n}}{n\pi^{2n}}\eta_{2n}$ is less than $\eta'\Sigma\frac{2^{2n}|z|^{2n}}{n\pi^{2n}}$, where ϵ', η' are the greatest values of ϵ_{2n}, η_{2n} respectively; hence

$$\log\frac{\sin z}{z} = -\Sigma\frac{z^{2n}}{n\pi^{2n}}S_{2n},$$

$$\log\cos z = \Sigma\frac{2^{2n}-1}{n\pi^{2n}}z^{2n}S_{2n}.$$

Since $S_{2n} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n$, we have the following infinite series for $\log\frac{\sin z}{z}$, $\log\cos z$,

$$\log\frac{\sin z}{z} = -2\frac{B_1}{1}\frac{z^2}{2!} - 2^3\frac{B_2}{2}\frac{z^4}{4!} - \ldots - 2^{2n-1}\frac{B_n}{n}\frac{z^{2n}}{(2n)!} - \ldots \quad \ldots(22),$$

where mod. $z < \pi$,

$$\log\cos z = -2(2^2-1)\frac{B_1}{1}\frac{z^2}{2!} - 2^3(2^4-1)\frac{B_2}{2}\frac{z^4}{4!} - \ldots$$
$$- 2^{2n-1}(2^{2n}-1)\frac{B_n}{n}\frac{z^{2n}}{(2n)!} - \ldots \quad \ldots(23),$$

where mod. $z < \frac{1}{2}\pi$.

The first few terms of the series (22), (23) are

$$\log\frac{\sin z}{z} = -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2835} - \ldots,$$

$$\log\cos z = -\frac{z^2}{2} - \frac{z^4}{12} - \frac{z^6}{45} - \ldots;$$

hence also

$$\log\tan z = \log z + \frac{z^2}{3} + \frac{7z^4}{90} + \frac{62z^6}{2835} + \ldots.$$

The series (22), (23) may be employed to calculate tables of logarithmic sines and cosines; it is best to calculate separately the first logarithms, $\log\left(1-\frac{z^2}{\pi^2}\right)$, $\log\left(1-\frac{4z^2}{\pi^2}\right)$, as we thus obtain the series in a more convergent form than in (22), (23).

We have

$$\log\sin\frac{m\pi}{2n}=\log\pi+\log\frac{m}{2n}+\log\left(1-\frac{m^2}{4n^2}\right)-\Sigma\left\{\left(\frac{B_r}{2r}\frac{\pi^{2r}}{(2r)!}-\frac{1}{2^{2r}r}\right)\frac{m^{2r}}{n^{2r}}\right\},$$

$$\log\cos\frac{m\pi}{2n}=\log\left(1-\frac{m^2}{n^2}\right)-\Sigma\left\{\left(\frac{2^{2r}-1}{2}\frac{B_r}{r}\frac{\pi^{2r}}{(2r)!}-\frac{1}{r}\right)\frac{m^{2r}}{n^{2r}}\right\}.$$

Multiplying the logarithms on the right-hand side of these equations by the modulus ·4342944819, we get the ordinary logarithms of $\sin\frac{m}{n}90°$, $\cos\frac{m}{n}90°$ to the base 10; the formulae thus found are

$L(\sin m/n\,90°)=$	$L(\cos m/n\,90°)=$
$\log m+\log(2n-m)+\log(2n+m)$	$\log(n-m)+\log(n+m)-2\log n$
$-3\log n+9{\cdot}594059885702190$	$+10{\cdot}000000000000000$
$-m^2/n^2\times{\cdot}070022826605901$	$-m^2/n^2\times{\cdot}101494859341892$
$-m^4/n^4\times{\cdot}001117266441661$	$-m^4/n^4\times{\cdot}003187294065451$
$-m^6/n^6\times{\cdot}000039229146453$	$-m^6/n^6\times{\cdot}000209485800017$
$-m^8/n^8\times{\cdot}000001729270798$	$-m^8/n^8\times{\cdot}000016848348597$
$-m^{10}/n^{10}\times{\cdot}000000084362986$	$-m^{10}/n^{10}\times{\cdot}000001480193986$
$-m^{12}/n^{12}\times{\cdot}000000004348715$	$-m^{12}/n^{12}\times{\cdot}000000136502272$
$-m^{14}/n^{14}\times{\cdot}000000000231931$	$-m^{14}/n^{14}\times{\cdot}000000012981715$
$-m^{16}/n^{16}\times{\cdot}000000000012659$	$-m^{16}/n^{16}\times{\cdot}000000001261471$
$-m^{18}/n^{18}\times{\cdot}000000000000702$	$-m^{18}/n^{18}\times{\cdot}000000000124567$
$-m^{20}/n^{20}\times{\cdot}000000000000039$	$-m^{20}/n^{20}\times{\cdot}000000000012456$
	$-m^{22}/n^{22}\times{\cdot}000000000001258$
	$-m^{24}/n^{24}\times{\cdot}000000000000128$
	$-m^{26}/n^{26}\times{\cdot}000000000000013$

These series were given by Euler, the decimals being given to twenty places.

EXAMPLES.

301. (1) *Find the values of* $\sum_1^\infty n^{-2}$, $\sum_1^\infty n^{-4}$, $\sum_1^\infty(2n-1)^{-2}$, $\sum_1^\infty(2n-1)^{-4}$.

We have

$$\log\frac{\sin x}{x}=\Sigma\log\left(1-\frac{x^2}{n^2\pi^2}\right)=-\frac{x^2}{\pi^2}\Sigma\frac{1}{n^2}-\frac{x^4}{2\pi^4}\Sigma\frac{1}{n^4}-\dots,$$

also $$\log\frac{\sin x}{x}=\log\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\dots\right)=-\left(\frac{x^2}{6}-\frac{x^4}{120}\right)-\frac{1}{2}\left(\frac{x^2}{6}\right)^2-\dots;$$

hence, equating the coefficients of x^2, x^4 in the two expressions for $\log\dfrac{\sin x}{x}$, we have $\Sigma n^{-2}=\frac{1}{6}\pi^2$, $\Sigma n^{-4}=\frac{1}{90}\pi^4$. Again

$$\log\cos x=\Sigma\log\left\{1-\frac{4x^2}{(2n-1)^2\pi^2}\right\}=-\frac{4x^2}{\pi^2}\Sigma\frac{1}{(2n-1)^2}-\frac{8x^4}{\pi^4}\Sigma\frac{1}{(2n-1)^4}-\dots;$$

and $$\log\cos x=\log\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\dots\right)=-\left(\frac{x^2}{2}-\frac{x^4}{24}\right)-\frac{1}{2}\left(\frac{x^2}{2}\right)^2,$$

therefore equating the coefficients of x^2 and x^4, we find

$$\Sigma(2n-1)^{-2}=\tfrac{1}{8}\pi^2,\quad \Sigma(2n-1)^{-4}=\tfrac{1}{96}\pi^4.$$

(2) *Sum the infinite series* $\dfrac{1}{1^2+x^2}+\dfrac{1}{3^2+x^2}+\dfrac{1}{5^2+x^2}+\dots$.

In the theorem (10), put $2z=ix\pi$; we thus find for the sum of the series, $\dfrac{\pi}{4x}\tanh\frac{1}{2}\pi x$. The sum might have been obtained directly from the expression for $\cosh\pi x$ in factors, by taking logarithms and differentiating.

(3) *Shew that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime is* $15/\pi^2$.

Let α, β, γ, ... denote the prime numbers 2, 3, 5, ..., then the required sum is equal to the infinite product

$$\left(1+\frac{1}{\alpha^2}\right)\left(1+\frac{1}{\beta^2}\right)\left(1+\frac{1}{\gamma^2}\right)\dots;$$

this is equal to $$\frac{\left(1-\frac{1}{\alpha^2}\right)^{-1}\left(1-\frac{1}{\beta^2}\right)^{-1}\left(1-\frac{1}{\gamma^2}\right)^{-1}\dots\dots}{\left(1-\frac{1}{\alpha^4}\right)^{-1}\left(1-\frac{1}{\beta^4}\right)^{-1}\left(1-\frac{1}{\gamma^4}\right)^{-1}\dots\dots}$$

or to $$\frac{\left(1+\frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\dots\right)\left(1+\frac{1}{\beta^2}+\frac{1}{\beta^4}+\dots\right)\left(1+\frac{1}{\gamma^2}+\frac{1}{\gamma^4}+\dots\right)\dots\dots}{\left(1+\frac{1}{\alpha^4}+\frac{1}{\alpha^8}+\dots\right)\left(1+\frac{1}{\beta^4}+\frac{1}{\beta^8}+\dots\right)\left(1+\frac{1}{\gamma^4}+\frac{1}{\gamma^8}+\dots\right)\dots\dots},$$

and this is equal to $$\frac{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots}{1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots}$$

or to $\dfrac{\frac{1}{6}\pi^2}{\frac{1}{90}\pi^4}$ which is equal to $15/\pi^2$.

(4) *An infinite straight line is divided by an infinite number of points into portions each of length* a. *Prove that if a point be taken such that* y *is its distance from the straight line, and* x *the projection on the straight line of its distance from one of the points of division, the sum of the squares of the reciprocals of the distances of this point from all the points of division is*

$$\frac{\pi}{\mathrm{a}\mathrm{y}}\,\frac{\sinh\dfrac{2\pi\mathrm{y}}{\mathrm{a}}}{\cosh\dfrac{2\pi\mathrm{y}}{\mathrm{a}}-\cos\dfrac{2\pi\mathrm{x}}{\mathrm{a}}}.$$

The series to be summed is $\sum_{-\infty}^{\infty} \frac{1}{y^2+(x+na)^2}$, which is equivalent to $\frac{1}{2iy}\sum_{-\infty}^{\infty}\left(\frac{1}{x-iy+na}-\frac{1}{x+iy+na}\right)$. The sum of the series is therefore

$$\frac{\pi}{2iya}\left\{\cot\frac{\pi(x-iy)}{a}-\cot\frac{\pi(x+iy)}{a}\right\},$$

or

$$\frac{\pi}{2iya}\cdot\frac{\sin\frac{2\pi iy}{a}}{\sin\frac{\pi(x+iy)}{a}\sin\frac{\pi(x-iy)}{a}},$$

which reduces to the given result.

EXAMPLES ON CHAPTER XVII.

1. Prove that

$$\cos(\tfrac{1}{2}\pi\sin\theta)=\tfrac{1}{4}\pi\cos^2\theta\left(1+\frac{\cos^2\theta}{2.4}\right)\left(1+\frac{\cos^2\theta}{4.6}\right)\ldots\ldots$$

2. Prove that

$$1+\sin x=\tfrac{1}{8}(\pi+2x)^2\left\{1-\frac{(\pi+2x)^2}{4^2\pi^2}\right\}^2\left\{1-\frac{(\pi+2x)^2}{8^2\pi^2}\right\}^2\ldots\ldots$$

3. Prove that $\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\frac{1}{(x+i)(x+j)}=-\pi^2$, where i, j have all unequal integral values, and x is not an integer.

Prove that

$$\frac{\left(\frac{\pi^2}{4}+1\right)\left(\frac{\pi^2}{4}+\frac{1}{9}\right)\left(\frac{\pi^2}{4}+\frac{1}{25}\right)\ldots\ldots}{\left(\frac{\pi^2}{4}+\frac{1}{4}\right)\left(\frac{\pi^2}{4}+\frac{1}{16}\right)\left(\frac{\pi^2}{4}+\frac{1}{36}\right)\ldots\ldots}=\frac{e^2+1}{e^2-1}.$$

5. Prove that

$$1+\frac{2x^2}{1+x^2}+\frac{2x^2}{2^2+x^2}+\frac{2x^2}{3^2+x^2}+\ldots\ldots=\frac{(1+4x^2)\left(1+\frac{4x^2}{3^2}\right)\left(1+\frac{4x^2}{5^2}\right)\ldots\ldots}{(1+x^2)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)\ldots\ldots}$$

6. Prove that

$$\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}+\ldots\ldots=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right).$$

7. If

$$\lambda(x)=x\prod_{1}^{\infty}\left\{1-\left(\frac{x}{na}\right)^2\right\},\quad \mu(x)=\prod_{1}^{\infty}\left\{1-\left(\frac{2}{2n-1}\cdot\frac{x}{a}\right)^2\right\},$$

express $\lambda(x+\tfrac{1}{2}a)$ in terms of $\mu(x)$, and $\mu(x+\tfrac{1}{2}a)$ in terms of $\lambda(x)$, and thence find the limit when m is infinite of $\frac{1.3.5\ldots(2m-1)}{2^m m!}\sqrt{2m+1}$.

8. If P_r denotes the products of $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$ taken r at a time, shew that $2^{2n}P_n = \frac{\pi^{2n}}{(2n)!} + \frac{\pi^{2n-2}}{(2n-2)!}P_1 + \frac{\pi^{2n-4}}{(2n-4)!}P_2 + \ldots\ldots + \frac{\pi^2}{2!}P_{n-1} + P_n$.

9. Prove that

$$1 - \frac{1^2}{2^2} - \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} - \ldots\ldots = \frac{2}{\pi}.$$

10. Sum the series

$$\frac{1}{1^4 . 3^4} + \frac{1}{3^4 . 5^4} + \frac{1}{5^4 . 7^4} + \ldots\ldots$$

11. Shew that the sum of the products of the fourth powers of the reciprocals of every pair of positive integers is $\frac{384\pi^8}{5!.9!}$.

12. Prove that

$$\left(1 + \frac{2}{1+1^2} + \frac{2}{1+2^2} + \frac{2}{1+3^2} + \ldots\ldots\right)\left(\frac{1}{4+1^2} + \frac{1}{4+3^2} + \frac{1}{4+5^2} + \ldots\ldots\right) = \frac{\pi^2}{8}.$$

13. Prove that the sum of the series

$$\left(\frac{1}{1.2.3}\right)^2 + \left(\frac{1}{2.3.4}\right)^2 + \left(\frac{1}{3.4.5}\right)^2 + \ldots\ldots$$

is $\frac{1}{4}\pi^2 - \frac{39}{16}$.

14. Shew that

$$\underset{r=\infty}{L} \frac{(m^2-1)(2^2m^2-1)\ldots\ldots(r^2m^2-1)}{\{m^2-(m-1)^2\}\{2^2m^2-(m-1)^2\}\ldots\ldots\{r^2m^2-(m-1)^2\}}$$

is $m-1$.

15. Shew that the sum of the series $\frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \ldots$ is $\frac{1}{4}\pi \operatorname{sech} \frac{1}{2}\pi x$.

16. Prove that

$$\tan^{-1} x - \tan^{-1} \tfrac{1}{3}x + \tan^{-1} \tfrac{1}{5}x - \ldots\ldots = \tan^{-1} \tanh \tfrac{1}{4}\pi x.$$

17. Prove that

$$\log 12 - 2 \log \pi = S_2 + \tfrac{1}{2}S_4 + \tfrac{1}{3}S_6 + \ldots\ldots + \frac{1}{n}S_{2n} + \ldots\ldots$$

where S_r is the sum of the reciprocals of the rth powers of all numbers which are not prime.

18. The side BC of a square $ABCD$ is produced indefinitely, and along it are measured CC_1, C_1C_2, C_2C_3, ... each equal to BC; if θ_1, θ_2, ... be the angles BAC_1, BAC_2, BAC_3, ..., shew that $\sin\theta_1 \sin\theta_2 \sin\theta_3 \ldots\ldots$ *ad inf.* $= \sqrt{2\pi} \operatorname{cosech} \pi$.

19. If 2, 3, 5, ... are all the prime numbers, shew that

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\ldots\ldots = 6/\pi^2$$

and
$$\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\frac{5^2}{5^2+1}\ldots\ldots=\pi^2/15,$$
$$\frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\ldots\ldots=\pi^4/105.$$
(*Euler.*)

20. Express the doubly infinite series $\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}(-1)^{m+n}\frac{\cos mx\cos ny}{mn(m^2+n^2)}$ in the form of a singly infinite series of cosines of multiples of y.

21. Prove that
$$\Pi\left\{\frac{(n\pi+a)^4+\beta^4}{n^4\pi^4}\right\}=(\sinh^2\beta\sqrt{2}+\cos^2\beta\sqrt{2}-2\cos 2a\cos\beta\sqrt{2}\cosh\beta\sqrt{2}$$
$$+\cos^2 2a)/4(a^4+\beta^4)$$
where n has all integral values, positive and negative, excluding zero.

22. Prove that
$$\frac{1}{1.2.3.4}+\frac{1}{5.6.7.8}+\frac{1}{9.10.11.12}+\ldots\ldots=\tfrac{1}{4}\log 2-\tfrac{1}{24}\pi,$$
$$\frac{1}{1.3.5.7}+\frac{1}{9.11.13.15}+\frac{1}{17.19.21.23}+\ldots\ldots=\frac{\pi}{96(2+\sqrt{2})}.$$

23. If $\phi(ix)=\left(1+\frac{ix}{a}\right)\left(1+\frac{ix}{b}\right)\left(1+\frac{ix}{c}\right)\ldots\ldots=A+iB$, shew that
$$\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\ldots\ldots=\tan^{-1}\frac{B}{A},$$
and hence shew that
$$\tan^{-1}\frac{x^2}{1^2}+\tan^{-1}\frac{x^2}{2^2}+\tan^{-1}\frac{x^2}{3^2}+\ldots\ldots=\tan^{-1}\left(\frac{\tan\frac{\pi x}{\sqrt{2}}-\tanh\frac{\pi x}{\sqrt{2}}}{\tan\frac{\pi x}{\sqrt{2}}+\tanh\frac{\pi x}{\sqrt{2}}}\right).$$

24. Prove that
$$\sum_1^\infty\frac{1}{n^4+x^4}=\frac{\pi\sqrt{2}}{4x^3}\,\frac{\sinh\pi x\sqrt{2}+\sin\pi x\sqrt{2}}{\cosh\pi x\sqrt{2}-\cos\pi x\sqrt{2}}-\frac{1}{2x^4}.$$

25. Prove that $\sum_{n=-\infty}^{n=\infty}\frac{1}{(n\pi+\theta)^2}=\operatorname{cosec}^2\theta.$

26. Prove that
$$\frac{e^{b+x}+e^{c-x}}{e^b+e^c}$$
$$=\left\{1+\frac{4(b-c)x+4x^2}{\pi^2+(b-c)^2}\right\}\left\{1+\frac{4(b-c)x+4x^2}{9\pi^2+(b-c)^2}\right\}\left\{1+\frac{4(b-c)x+4x^2}{25\pi^2+(b-c)^2}\right\}\ldots\ldots$$
and
$$\frac{e^{b+x}-e^{c-x}}{e^b-e^c}=\left(1+\frac{2x}{b-c}\right)\left\{1+\frac{4(b-c)x+4x^2}{4\pi^2+(b-c)^2}\right\}\left\{1+\frac{4(b-c)x+4x^2}{16\pi^2+(b-c)^2}\right\}\ldots\ldots$$
(*Euler.*)

27. If

$$P=\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\frac{1}{5n-m}-\frac{1}{5n+m}+\ldots\ldots,$$

$$Q=\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(3n-m)^2}+\frac{1}{(3n+m)^2}+\ldots\ldots,$$

$$R=\frac{1}{(n-m)^3}-\frac{1}{(n+m)^3}+\frac{1}{(3n-m)^3}-\frac{1}{(3n+m)^3}+\ldots\ldots$$

$$S=\frac{1}{(n-m)^4}+\frac{1}{(n+m)^4}+\frac{1}{(3n-m)^4}+\frac{1}{(3n+m)^4}+\ldots\ldots,$$

prove that

$$P=\frac{k\pi}{2n},\quad Q=\frac{(2k^2+2)\,\pi^2}{2.4.n^2},\quad R=\frac{(6k^3+6k)\,\pi^3}{2.4.6.n^3},\quad S=\frac{(24k^4+32k^3+8)\,\pi^4}{2.4.6.8.n^4},$$

where $$k=\tan\frac{m\pi}{2n}.$$ (*Euler.*)

28. Prove that the sum of the series $1-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\ldots\ldots$, in which all odd numbers not divisible by 3 are taken, is $\pi^3/18\sqrt{3}$. (*Euler.*)

29. Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by 3 is $4\pi^2/27$. (*Euler.*)

30. Prove that

$$\frac{\sinh y+\sinh c}{\sinh c}=\left(1+\frac{y}{c}\right)\left(1-\frac{2cy-y^2}{\pi^2+c^2}\right)\left(1+\frac{2cy+y^2}{4\pi^2+c^2}\right)\left(1-\frac{2cy-y^2}{9\pi^2+c^2}\right)\ldots\ldots$$

and

$$\frac{\cosh y-\cosh c}{1-\cosh c}=\left(1-\frac{y^2}{c^2}\right)\left(1-\frac{2cy-y^2}{4\pi^2+c^2}\right)\left(1+\frac{2cy+y^2}{4\pi^2+c^2}\right)\left(1-\frac{2cy-y^2}{16\pi^2+c^2}\right)\ldots\ldots$$

(*Euler.*)

31. Prove that when n is odd

$$\cot^2\frac{2\pi}{2n}+\cot^2\frac{4\pi}{2n}+\ldots\ldots+\cot^2\frac{(n-1)\pi}{2n}=\tfrac{1}{6}(n-1)(n-2),$$

$$\cot^4\frac{2\pi}{2n}+\cot^4\frac{4\pi}{2n}+\ldots\ldots+\cot^4\frac{(n-1)\pi}{2n}=\tfrac{1}{90}(n-1)(n-2)(n^2+3n-13).$$

32. Prove that the infinite product $(1+x^{2n})\left(1+\frac{x^{2n}}{3^{2n}}\right)\left(1+\frac{x^{2n}}{5^{2n}}\right)\ldots\ldots$ is equal to

$$\frac{1}{2^{\frac{1}{2}n}}\prod_{1}^{n-1}(\cosh\pi\alpha x+\cos\pi\beta x),\ \text{or}\ \frac{1}{2^{\frac{1}{2}(n-1)}}\cosh\tfrac{1}{2}\pi x\prod_{1}^{n-2}(\cosh\pi\alpha x+\cos\pi\beta x),$$

according as n is even or odd, α_r, β_r denoting $\sin\frac{r\pi}{2n}$, $\cos\frac{r\pi}{2n}$ respectively, where r is an odd number. (*Glaisher.*)

33. Prove that the infinite product $x^n(1+x^{2n})\left(1+\frac{x^{2n}}{2^{2n}}\right)\left(1+\frac{x^{2n}}{3^{2n}}\right)\ldots\ldots$ is equal to

$$\frac{1}{2^{\frac{1}{2}n}\pi^n}\prod_1^{n-1}(\cosh 2\pi\alpha x-\cos 2\pi\beta x),\text{ or }\frac{1}{2^{\frac{1}{2}(n-1)}\pi^n}\sinh\pi x\prod_1^{n-2}(\cosh 2\pi\alpha x-\cos 2\pi\beta x),$$

according as n is even or odd, α, β having the same meaning as in the last question. (*Glaisher.*)

34. Prove that

$$\frac{1}{1^{2n}+x^{2n}}+\frac{1}{2^{2n}+x^{2n}}+\frac{1}{3^{2n}+x^{2n}}+\ldots\ldots$$
$$=\frac{\pi}{nx^{2n-1}}\sum_1^{n-1}\frac{\alpha\sinh 2\pi\alpha x+\beta\sin 2\pi\beta x}{\cosh 2\pi\alpha x-\cos 2\pi\beta x}-\frac{1}{2x^{2n}},$$

α, β having the same meaning as in the last question. (*Glaisher.*)

35. Shew that

$$\frac{ax+by}{x^2+y^2}+\sum_{r=1}^{r=\infty}\left\{\frac{ax+by+r(a^2+b^2)}{(x+ra)^2+(y+rb)^2}+\frac{ax+by-r(a^2+b^2)}{(x-ra)^2+(y-rb)^2}\right\}$$

is equal to

$$\pi\sin\left(2\pi\frac{ax+by}{a^2+b^2}\right)\Big/\left\{\cosh\left(2\pi\frac{ay-bx}{a^2+b^2}\right)-\cos\left(2\pi\frac{ax+by}{a^2+b^2}\right)\right\}.$$

CHAPTER XVIII.

CONTINUED FRACTIONS.

Proof of the irrationality of π.

302. LET $f(c)$ denote the sum of the convergent series

$$1-\frac{x^2}{1.c}+\frac{x^4}{1.2.c(c+1)}-\frac{x^6}{1.2.3.c(c+1)(c+2)}+\ldots,$$

then
$$f(c+1)-f(c)=\frac{x^2}{c(c+1)}f(c+2);$$

hence
$$\frac{f(c)}{f(c+1)}=1-\frac{x^2}{c(c+1)}\,\frac{f(c+2)}{f(c+1)},$$

therefore $f(c+1)/f(c)$ can be expressed as a continued fraction of the second class

$$\frac{1}{1-}\,\frac{x^2/c(c+1)}{1-}\,\frac{x^2/(c+1)(c+2)}{1-}\,\frac{x^2/(c+2)(c+3)}{1-}\ldots.$$

Let $c=\frac{1}{2}$, and write $\frac{1}{2}x$ for x, the series $f(c)$ becomes

$$1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\ldots$$

or $\cos x$, and $f(c+1)$ becomes $\dfrac{\sin x}{x}$;

hence
$$\frac{\tan x}{x}=\frac{1}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\frac{x^2}{7-}\ldots,$$

an expression for $\tan x$ as a continued fraction of the second class.

303. Lambert's proof[1] of the irrationality of π depends on the continued fraction found in the last Article. Put $x=\frac{1}{4}\pi$, and if possible let $\frac{1}{4}\pi=m/n$, where m and n are integers; we have then

$$1=\frac{m}{n-}\,\frac{m^2}{3n-}\,\frac{m^2}{5n-}\,\frac{m^2}{7n-}\ldots;$$

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[1] Published in the memoirs of the Academy of Berlin in 1761.

now after a certain term, the denominators of the fractions m/n, $m^2/3n$, $m^2/5n$, ... exceed the numerators by a number greater than unity, hence, by a well-known theorem[1], the continued fraction on the right-hand side of the equation has an irrational limit, and cannot therefore be equal to unity. Hence $\frac{1}{4}\pi$ cannot be equal to a fraction m/n in which m and n are integers, and therefore π is irrational. This result is of course included in the much wider theorem of Art. 251(3), that π is a transcendental number.

Transformation of the quotient of two hypergeometric series.

304. The fraction $F(\alpha, \beta+1, \gamma+1, x)/F(\alpha, \beta, \gamma, x)$, where $F(\alpha, \beta, \gamma, x)$ denotes the hypergeometrical series

$$1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha+1) . \beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2 + \dots,$$

can be transformed into the continued fraction

$$\frac{1}{1-}\,\frac{k_1 x}{1-}\,\frac{k_2 x}{1-}\,\frac{k_3 x}{1-}\dots,$$

where

$$k_1 = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)},\quad k_2 = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)},\quad k_3 = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)}$$

$$k_4 = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}, \dots\ k_{2n-1} = \frac{(\alpha+n-1)(\gamma+n-1-\beta)}{(\gamma+2n-2)(\gamma+2n-1)},$$

$$k_{2n} = \frac{(\beta+n)(\gamma+n-\alpha)}{(\gamma+2n-1)(\gamma+2n)}.$$

As an example of the use of this transformation, taking the series

$$\phi = \sin\phi\cos\phi\left\{1 + \frac{2}{3}\sin^2\phi + \frac{2 . 4}{3 . 5}\sin^4\phi + \dots\right\},$$

and putting $\alpha = 1$, $\beta = 0$, $\gamma = \frac{1}{2}$, $x = \sin^2\phi$ in the above formula of transformation, we find

$$\phi = \frac{\sin\phi\cos\phi}{1-}\,\frac{\frac{1 . 2}{1 . 3}\sin^2\phi}{1-}\,\frac{\frac{1 . 2}{3 . 5}\sin^2\phi}{1-}\,\frac{\frac{3 . 4}{5 . 7}\sin^2\phi}{1-}\dots.$$

The second convergent gives Snellius' formula for ϕ,

$$\phi = \frac{\sin\phi\cos\phi}{1-\frac{2}{3}\sin^2\phi} = \frac{3\sin 2\phi}{2(2+\cos 2\phi)}.$$

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[1] See Chrystal's *Algebra*, Vol. II. p. 484.

Euler's Transformation.

305. Other series may be transformed by means of Euler's theorem[1]

$$u_1+u_2+u_3+u_4+\dots=\frac{u_1}{1-}\,\frac{u_2}{u_1+u_2-}\,\frac{u_1u_3}{u_2+u_3-}\,\frac{u_2u_4}{u_3+u_4-}\dots,$$

which may also be written in the form

$$\frac{1}{a_1}\pm\frac{1}{a_2}\pm\frac{1}{a_3}\pm\frac{1}{a_4}\pm\dots=\frac{1}{a_1\mp}\,\frac{a_1^2}{a_2\pm a_1\mp}\,\frac{a_2^2}{a_3\pm a_2\mp}\dots.$$

As an example of this method, we obtain from the theorem

$$\frac{\pi}{n}\cot\frac{m\pi}{n}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\dots$$

the theorem

$$\frac{\pi}{n}\cot\frac{m\pi}{n}=\frac{1}{m+}\,\frac{m^2}{n-2m+}\,\frac{(n-m)^2}{2m+}\,\frac{(n+m)^2}{2m+}\,\frac{(2n-m)^2}{2m+}\,\frac{(2n+m)^2}{n-2m+}\dots.$$

EXAMPLES ON CHAPTER XVIII.

Investigate the theorems in Examples (1) to (13).

1. $\dfrac{\tanh x}{x}=\dfrac{1}{1+}\,\dfrac{x^2}{3+}\,\dfrac{x^2}{5+}\dots\dots$

2. $\tan nx=\dfrac{n\tan x}{1-}\,\dfrac{(n^2-1)\tan^2 x}{3-}\,\dfrac{(n^2-4)\tan^2 x}{5-}\,\dfrac{(n^2-9)\tan^2 x}{7-}\dots\dots$

when $x<\frac{1}{2}\pi$, n being unrestricted.

3. $\tan nx=\dfrac{n\tan x}{1-\tan^2 x-}\,\dfrac{(n^2-4)\tan^2 x}{3-3\tan^2 x-}\,\dfrac{(n^2-16)\tan^2 x}{5-5\tan^2 x-}\dots\dots$

4. $\tan nx=\dfrac{n\tan x}{1-}\,\dfrac{(n^2-1)\tan^2 x}{3-\tan^2 x-}\,\dfrac{(n^2-9)\tan^2 x}{5-3\tan^2 x-}\dots\dots$

5. $\tan^{-1}x=\dfrac{x}{1+}\,\dfrac{x^2}{3+}\,\dfrac{4x^2}{5+}\dots\dots$

6. $\tan^{-1}x=\dfrac{x}{1-x^2+}\,\dfrac{4x^2}{3-3x^2+}\,\dfrac{16x^2}{5-5x^2+}\dots\dots$

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[1] See Chrystal's *Algebra*, Vol. II. p. 487.

7. $$\tan^{-1} x = \frac{x}{1+}\,\frac{x^2}{3-x^2+}\,\frac{9x^2}{5-3x^2+}\dots\dots$$

8. $$\tan nx = \frac{n \tanh x}{1-}\,\frac{(n^2+1)\tanh^2 x}{3-}\,\frac{(n^2+4)\tanh^2 x}{5-}\dots\dots$$

9. $$\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} = 1+\frac{1}{n-1+}\,\frac{(n-1)n}{1+}\,\frac{n(n+1)}{n-1+}\,\frac{(2n-1)2n}{1+}\dots\dots$$

10. $$\frac{\sin \pi x}{\pi x} = 1+\frac{x}{1-}\,\frac{1+x}{x-}\,\frac{1-x}{1+x-}\,\frac{2(2+x)}{x-}\,\frac{2(2-x)}{1+x-}\dots\dots$$

11. $$\cos\frac{\pi x}{2} = 1+\frac{x}{1-}\,\frac{1+x}{x-}\,\frac{1-x}{2+x-}\,\frac{3(3+x)}{x-}\dots\dots$$

12. $$\cot\frac{1}{x} = \frac{1}{x-1+}\,\frac{1}{1+}\,\frac{1}{3x-2+}\,\frac{1}{1+}\,\frac{1}{5x-2+}\dots\dots$$

13. $$1-\frac{\sin\theta}{\theta} = \frac{\frac{1.2}{1.3}\sin^2\tfrac{1}{2}\theta}{1-}\,\frac{\frac{1.2}{3.5}\sin^2\tfrac{1}{2}\theta}{1-}\,\frac{\frac{3.4}{5.7}\sin^2\tfrac{1}{2}\theta}{1-}\,\frac{\frac{3.4}{7.9}\sin^2\tfrac{1}{2}\theta}{1-}\dots\dots$$

MISCELLANEOUS EXAMPLES.

1. Prove that if m is a positive integer

$$\frac{\cos mx - \cos ma}{\cos x - \cos a} = \operatorname{cosec} a \{2 \sin a \cos (m-1)x + 2 \sin 2a \cos (m-2)x + \ldots\ldots$$
$$+ 2 \sin (m-1) a \cos x + \sin ma\}. \qquad \text{(\textit{Hermite.})}$$

2. Prove that if m and n are positive integers

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n} \Sigma (-1)^k \sin ma \cot \frac{x-a}{2},$$

where $a = \frac{k\pi}{n}$, and that the expressions are also equal to

$$\frac{1}{2n} \Sigma (-1)^k \sin ma \cot (x-a),$$

or

$$\frac{1}{2n} \Sigma (-1)^k \sin ma \operatorname{cosec} (x-a),$$

according as $m+n$ is even or odd. (*Hermite.*)

3. Prove that

$$\cot (x-a) \cot (x-\beta) \ldots\ldots \cot (x-\lambda) = \cos \tfrac{1}{2} n\pi + \Sigma A \cot (x-a),$$

where $A = \cot (a-\beta) \cot (a-\gamma) \ldots\ldots \cot (a-\lambda)$. (*Hermite.*)

4. If A, B, C be the angles of a triangle, and x, y, z are real quantities determined by the equations

$$\cosh x (\sin B \sin C)^{\frac{1}{2}} = \cos \tfrac{1}{2} A,$$

$$\cosh y (\sin C \sin A)^{\frac{1}{2}} = \cos \tfrac{1}{2} B, \quad \cosh z (\sin A \sin B)^{\frac{1}{2}} = \cos \tfrac{1}{2} C,$$

then any three points so situated that the distances between each pair are proportional to x, y, z, respectively, lie on a straight line.

5. If $x > \frac{1}{2}$, shew that $\tan \frac{1}{1+x^2} > \frac{1}{1+x+x^2}$, and $< \frac{1}{1-x+x^2}$.

6. Prove that $\frac{1}{n} \sum\limits_{p=1}^{p=m} \sum\limits_{k=0}^{k=n-1} \frac{2pk\pi}{n}$ is equal to the greatest integer in m/n.

7. Prove that

$$\tan^{-1}\frac{4b^2}{(2a+b)^2+3b^2}+\tan^{-1}\frac{4b^2}{(2a+3b)^2+3b^2}+\ldots\ldots+\tan^{-1}\frac{4b^2}{(2a+\overline{2n-1}b)^2+3b^2}$$

is equal to $\tan^{-1}\frac{nb^2}{a^2+nab+b^2}$; and hence shew that the sum of the infinite series $\cot^{-1}(1^2+\frac{3}{4})+\cot^{-1}(2^2+\frac{3}{4})+\cot^{-1}(3^2+\frac{3}{4})+\ldots\ldots$ is $\cot^{-1}\frac{1}{2}$.

8. If $$\tan A\sec B+\tan B\sec A=\tan C,$$
prove that
$$\tan A\sec A+\tan B\sec B+\tan C\sec C+2\tan A\tan B\tan C=0.$$

Trace a connection between this result and the known theorem that
$$\sin A\cos A+\sin B\cos B+\sin C\cos C-2\sin A\sin B\sin C=0,$$
where A, B, C are the angles of a triangle.

9. If m and n be any numbers, prove that

$$\sin x\left\{1-\frac{n(n+1)}{(m+n)(m+n+1)}\frac{x^2}{2!}\right.$$
$$\left.+\frac{n(n+1)(n+2)(n+3)}{(m+n)(m+n+1)(m+n+2)(m+n+3)}\frac{x^4}{4!}-\ldots\ldots\right\}$$
$$=(m+n\cos x)\frac{1}{m+n}\cdot\frac{x}{1}$$
$$-\{m(m+1)(m+2)+n(n+1)(n+2)\cos x\}\frac{1}{(m+n)(m+n+1)(m+n+2)}\frac{x^3}{3!}+\ldots.$$

10. Prove that

$$\begin{vmatrix} 1 & \cos\alpha, & \cos(\alpha+\beta), & \cos(\alpha+\beta+\gamma), & \cos(\alpha+\beta+\gamma+\delta) \\ \cos\alpha, & 1 & \cos\beta, & \cos(\beta+\gamma), & \cos(\beta+\gamma+\delta) \\ \cos(\alpha+\beta), & \cos\beta, & 1 & \cos\gamma, & \cos(\gamma+\delta) \\ \cos(\alpha+\beta+\gamma), & \cos(\beta+\gamma), & \cos\gamma, & 1 & \cos\delta \\ \cos(\alpha+\beta+\gamma+\delta), & \cos(\beta+\gamma+\delta), & \cos(\gamma+\delta), & \cos\delta, & 1 \end{vmatrix}=0.$$

11. Prove that the determinant

$$\begin{vmatrix} 1, & \cos A, & \sin A, & \cos(3A+X) \\ 1, & \cos B, & \sin B, & \cos(3B+X) \\ 1, & \cos C, & \sin C, & \cos(3C+X) \\ 1, & \cos D, & \sin D, & \cos(3D+X) \end{vmatrix}$$

is equal to $2\sin(A+S+X)$ multiplied by the product of the sines of half the differences between A, B, C, D, and also by a numerical factor, S denoting
$$\tfrac{1}{2}(A+B+C+D).$$

12. Prove that, if

$$\cos(4x-y-z)\sin(y-z)+\cos(4y-z-x)\sin(z-x)+\cos(4z-x-y)\sin(x-y)=0,$$

and no two of the three x, y, z are equal, or differ by a multiple of π, then
$$\cos 2x+\cos 2y+\cos 2z=0.$$

13. Prove that, if γ and δ be two values of θ between 0 and π, which satisfy the equation

$$\sin 2\theta \cos^2(\alpha+\beta)+\sin 2\alpha \cos^2(\beta+\theta)+\sin 2\beta \cos^2(\alpha+\theta)=0,$$

then α and β satisfy the equation

$$\sin 2\phi \cos^2(\gamma+\delta)+\sin 2\gamma \cos^2(\delta+\phi)+\sin 2\delta \cos^2(\gamma+\phi)=0.$$

14. If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the three values of $\tan\frac{\theta}{3}$ obtained when $\tan\theta$ is given, prove that

(1) $\cos\alpha\cos\beta\cos\gamma\sin(\alpha+\beta+\gamma)+\sin\alpha\sin\beta\sin\gamma\cos(\alpha+\beta+\gamma)=0.$

(2) $\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)=\sin 2\alpha\sin 2\beta\sin 2\gamma.$

15. Shew that

$$\frac{\Sigma\sin(\beta-\gamma)\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\sin\frac{2\alpha+3\beta+3\gamma}{2}}{\Sigma\sin(\beta-\gamma)\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\cos\frac{2\alpha+3\beta+3\gamma}{2}}$$

$$=\frac{\sin 2(\alpha+\beta+\gamma)+\Sigma\sin(2\alpha+\beta+\gamma)}{\cos 2(\alpha+\beta+\gamma)+\Sigma\cos(2\alpha+\beta+\gamma)},$$

where the summation Σ refers to the sum formed by a cyclical interchange of the angles α, β, γ.

16. Prove that, if

$$u=1+\frac{2\cos\theta}{1+}\,\frac{2\cos\frac{\theta}{2}}{1+}\,\frac{2\cos\frac{\theta}{2^2}}{1+}\,\ldots\ldots,$$

the error made in taking the nth convergent to u instead of u is

$$\frac{2(u^2-1)}{u-\sqrt{4-u^2}\cot\frac{\cos^{-1}\frac{1}{2}u}{(-2)^n}}.$$

17. Prove that the series

$$\frac{1}{n^2-1}-\frac{1}{3n^2-3}+\frac{1}{5n^2-5}-\ldots\ldots\text{ to }\infty$$

has for its sum $\quad\frac{\pi}{4}\left\{\sec\frac{\pi}{2n}-1\right\}.$

18. Shew that the equation $\tan z=az$, where a is real, cannot have imaginary roots unless $a<1$, and that then it has one pair of imaginary roots.

19. Shew that the antiparallels through A, B, C to any three lines AO, BO, CO with respect to the angles A, B, C of the triangle ABC meet in a point O', and that the six feet of the perpendiculars from O and O' on the sides lie on a circle.

If GL, GM, GN be perpendiculars to the sides BC, CA, AB from the centroid G, and P any point on the circumference of the circle LMN, shew that

$$(4a^2+b^2+c^2)AP^2+(a^2+4b^2+c^2)BP^2+(a^2+b^2+4c^2)CP^2$$

is constant.

20. If x be real, and $1>x>0$, and if $\tan^{-1} z$ mean the least positive angle whose tangent is z, shew that

$$\sum_{r=0}^{r=\infty} (-1)^r \tan^{-1} \frac{(2r+1)\,x}{(2r+1)^2-x^2} = \tan^{-1}\left\{\sinh\frac{\pi x}{3}\sec\frac{\pi x\sqrt{3}}{4}\right\}.$$

21. If P be any point on a circle passing through the centres of the three circles escribed to the triangle ABC, prove the relation

$$\frac{AP^2}{bc}(1+\cos A-\cos B-\cos C)+\frac{BP^2}{ca}(1-\cos A+\cos B-\cos C)$$
$$+\frac{CP^2}{ab}(1-\cos A-\cos B+\cos C)=1+\cos A+\cos B+\cos C.$$

22. If $u_n=A\cos n\theta+B\sin n\theta$, where A and B are independent of n, prove geometrically the equation

$$u_{n+1}-2u_n\cos\theta+u_{n-1}=0.$$

Prove that

$$\frac{2^6\sin^7\theta+\sin 7\theta}{2^6\cos^7\theta-\cos 7\theta}=\tan\theta\tan^2\left(\theta+\frac{\pi}{6}\right)\tan^2\left(\theta-\frac{\pi}{6}\right).$$

23. If O_1, O_2; G_1, G_2; N_1, N_2; P_1, P_2 be respectively the two positions of the circumcentre, centroid, nine-points centre, and orthocentre of a triangle in the ambiguous case, prove that

$$2O_1O_2=3G_1G_2\operatorname{cosec} A=4N_1N_2=P_1P_2\sec A;$$

a, b, A being the given parts.

24. Lines $AB'C'$, $BC'A'$, $CA'B'$ are drawn through the angular points A, B, C of a triangle, making equal angles θ with AB, BC, CA respectively; and lines $AC''B''$, $CB''A''$, $BA''C''$ making equal angles θ with AC, CB, BA respectively. Shew that the triangles $A'B'C'$, $A''B''C''$ are equal in all respects, the area of each being $\Delta\sin^2\theta(\cot\theta-\cot A-\cot B-\cot C)^2$. Shew also that if T_A', T_A'' be the tangents to the circumcircles of these triangles from the point A, with a similar notation for the tangents from B and C, then will

$$aT_A'=cT_C'',\quad bT_B'=aT_A'',\quad cT_C'=bT_B''$$

25. Sum the series

$$\sum_{n=-q}^{n=p}\left[\frac{1}{(-1)^n x-a-n}+\frac{1}{n}\right],$$

where the value $n=0$ is omitted, and p, q are positive integers to be increased without limit.

26. Shew that, if $a=2\pi/17$, the quantities

$$\cos a+\cos 3^2 a+\cos 3^4 a+\cos 3^6 a,\ \text{and}\ \cos 3a+\cos 3^5 a+\cos 3^5 a+\cos 3^7 a$$

are the roots of the equation $z^2+\frac{1}{2}z=1$, and explain how the process thus indicated can be continued to obtain the value of $\cos a$.

A, B, C, D, E, F, G, H, K are nine consecutive vertices of a regular polygon of seventeen sides inscribed in a circle whose centre is O; a, β, γ, δ are the projections upon OA of the middle points of the chords BE, CK, DF, GH respectively; shew that the common chord of the two circles on $\alpha\beta$ and $\gamma\delta$ as diameters passes through O, and is of length $\frac{1}{2}OA$.

27. If α, β, γ, δ be the distances of the nine-points centre from those of the inscribed and escribed circles of a triangle ABC, shew that

$$\frac{1}{\beta+\gamma+\delta-11\alpha}+\frac{1}{\gamma+\delta+\alpha-11\beta}+\frac{1}{\delta+\alpha+\beta-11\gamma}+\frac{1}{\alpha+\beta+\gamma-11\delta}=0,$$

and that $$\alpha^2+\beta^2+\gamma^2+\delta^2=R^2(13-8\cos A\cos B\cos C),$$

where R is the radius of the circumcircle.

28. Prove that $$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}.$$

29. Prove that if I be the centre of the inscribed circle of a triangle ABC, and L, M, N the centres of the escribed circles, the circles inscribed in the triangles IMN, INL, ILM touch the circle ABC, and the tangents of the angles of the triangle formed by the three points of contact are respectively equal to

$$\frac{2\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C-\sin\frac{1}{2}B-\sin\frac{1}{2}C-2}{1-\cos\frac{1}{2}B-\cos\frac{1}{2}C+\sin\frac{1}{2}B+\sin\frac{1}{2}C}$$

and two similar expressions.

30. Shew that if x be not an integer, the series

$$\Sigma\,\frac{2x+m+n}{(x+m)^2(x+n)^2},$$

in which m and n receive in every possible way unequal values, zero or integers lying between I and $-I$, vanishes when I increases indefinitely.

31 Shew that $\sin^m\theta\cos^n\theta$ can be expanded in the form

$$A_0\genfrac{}{}{0pt}{}{\sin}{\cos}(m+n)\theta+A_1\genfrac{}{}{0pt}{}{\sin}{\cos}(m+n-2)\theta+A_2\genfrac{}{}{0pt}{}{\sin}{\cos}(m+n-4)\theta+\&c.$$

when m and n are positive integers.

Shew also that

$$(p+2)A_{p+2}+(m-n)A_{p+1}+(m+n-p)A_p=0,$$

except in the case of the last terms of the series, when both m and n are even.

32. The circumference of a circle whose centre is O is divided into n equal parts at the points P_1, P_2, P_3, ... P_n, and Q is any internal point. Prove that

$$\tan P_1QO+\tan P_2QO+\ldots\ldots+\tan P_nQO=n\tan P'Q'O,$$

where P' is a point on the circle such that $QOP'=n\,.\,QOP_1$, and Q' is a point on QO such that (if the ordinates QR, $Q'R'$ cut the circle in R, R')

$$QOR'=n\,.\,QOR.$$

33. Prove that, if $m_1, m_2, \ldots\ldots m_s$ are the integers less than and prime to m, and if $p_1, p_2, \ldots\ldots$ are the *different* prime factors of m,

$$\prod_1^s \sin\left(\theta+\frac{m_r\pi}{m}\right)=\frac{\sin m\theta\,.\,\Pi\sin\dfrac{m\theta}{p_1p_2}\,.\,\Pi\sin\dfrac{m\theta}{p_1p_2p_3p_4}\ldots\ldots}{2^s\,\Pi\sin\dfrac{m\theta}{p_1}\,.\,\Pi\sin\dfrac{m\theta}{p_1p_2p_3}\ldots\ldots}.$$

34. Prove that the sum of the products

$$\sin pa\sin q\left(a+\frac{2\pi}{3}\right)\sin r\left(a+\frac{4\pi}{3}\right)$$

for all positive integral values of p, q, r which are such that $p+q+r=s$, when $s \geqq 3$, is zero unless s is a multiple of 3, and is $-\frac{1}{4}\sin sa$, when s is a multiple of 3.

35. Prove that

$$\tan\theta=\frac{x}{2}\left\{1-\frac{x^2}{4}+\frac{x^4}{8}-\frac{5}{64}x^6+\ldots\ldots\right\},$$

$$\sin\theta=\frac{x}{2}\left\{1-\frac{3}{8}x^2+\frac{31}{128}x^4-\frac{187}{1024}x^6+\ldots\ldots\right\},$$

$$2\sin\tfrac{1}{2}\theta=\frac{x}{2}\left\{1-\frac{11}{32}x^2+\frac{431}{2048}x^4-\ldots\ldots\right\},$$

where $x=\tan 2\theta$.